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ERRATUM TO: A FIBERING THEOREM FOR 3-MANIFOLDS

JORDAN A. SAHATTCHIEVE

Formerly Department of Mathematics, University of Michigan in Ann Arbor
e-mail address: jantonov@umich.edu

Abstract. Erratum to the paper titled *A fibering theorem for 3-manifolds*, which appeared in print in the journal of Groups, Complexity, Cryptology Volume 13 Issue 2, published Nov. 11, 2021.

1. Overview

Since the publication of the article, the author noticed an error in the proof of Case 2 of Theorem 4.6, a correct but incorrectly proved claim in the proof of the Main Theorem, and certain minor errors and misprints. The author addresses the two substantive errors in Sections 2 and 3 below.

2. Correction to the proof of Case 2 of Theorem 4.6

In Case 2 of the proof of Theorem 4.6, the author made the mistake of assuming part of the claim they are proving - namely, they have incorrectly assumed that the surface T, which is referenced on line 3 from the bottom on page 11 of the article, is the torus. Proving the theorem was possible under a somewhat more restrictive hypothesis. Below is a restatement of Theorem 4.6 and its proof under the revised hypothesis. This change in Theorem 4.6 requires that the hypotheses and proofs of all the subsequent results which rely on Theorem 4.6, including the definitions of property (A) and property (A'), be changed in the obvious way.

Theorem 4.6. *Let M be a compact 3-manifold with $\pi_1(M) = G$, and suppose that M splits along an incompressible torus $\mathcal{T}$, $M = X_1 \cup_{\mathcal{T}} X_2$, or $M = X_1 \cup_{\mathcal{T}}$. Suppose that:*

(1) *G contains a nontrivial, subnormal subgroup $N = N_0 \triangleleft ... \triangleleft N_{n-1} \triangleleft N_n = G$ such that $N \neq \mathbb{Z}$ and,*
(2) *at least n terms in the subnormal series $N = N_0 \triangleleft ... \triangleleft N_{n-1} \triangleleft N_n = G$ are finitely generated and,*
(3) *either all inclusions $N_i \hookrightarrow N_{i+1}$, for $i > 0$, are of finite index, or there exist indices i_0 and i_1, $i_0 \neq i_1$, $i_0, i_1 > 1$, such that the inclusions $N_i \hookrightarrow N_{i+1}$ are of infinite index for $i = i_0, i_1$, or $N = N_0$ is finitely generated and there exists a value of the index i for which the inclusion $N_i \hookrightarrow N_{i+1}$ is of infinite index and,*

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(4) *G contains a finitely generated subgroup U of infinite index in G such that $N < U$.*
If the graph of groups $\mathcal{U}$ corresponding to U has infinite diameter, then M is finitely covered by a torus bundle over $\mathbb{S}^1$ with fiber T, and U and $\pi_1(T)$ are commensurable.

Proof. ...

Case 2: $|N_i : N_{i-1}| = \infty$ for $i = i_1, i_2 > 1$, where $i_1 \neq i_2$

Let i_0 be the largest integer index for which $|N_{i_0} : N_{i_0-1}| = \infty$. In this case, we consider the finite cover $M_{N_{i_0}}$ whose fundamental group is N_{i_0}. Since N_{i_0-1} is assumed to be finitely generated, it is also finitely presented by Theorem 2.1 in [3]. Now, by Theorem 1.2 either $M_{N_{i_0}}$ itself or, in the case when $M_{N_{i_0}}$ is a union of two twisted I-bundles, a finite cover of $M_{N_{i_0}}$ fibers over $\mathbb{S}^1$ with fiber a compact surface F and, further, N_{i_0-1} is a subgroup of finite index in $\pi_1(F)$. Since there are at least two distinct values of the index i, for which $|N_i : N_{i-1}| = \infty$, there is a value of the index $1 < k < i_0$, for which $|N_k : N_{k-1}| = \infty$. Consider now the finite cover $F_{N_{i_0-1}}$ of F, whose fundamental group is N_{i_0-1}. Since N_{k-1} is assumed to be finitely generated, applying Theorem 2.1 with $N = N_0$ and $U = N_{k-1}$, we conclude that $F_{N_{i_0-1}}$ is the torus or the Klein bottle. Hence, F itself is the torus or the Klein bottle. However, because $M_{N_{i_0}}$, being a finite cover of the orientable manifold M, is itself an orientable 3-manifold, while all four Klein bottle bundles over $\mathbb{S}$ are non-orientable 3-manifolds, we conclude that F is the torus T and that N_{i_0-1} is a subgroup of finite index in $\pi_1(T)$, which is a group isomorphic to $\mathbb{Z}^2$.

Finally, we show that U is commensurable with $\pi_1(T)$. Consider $U \cap N_{i_0-1}$; this group is a subgroup of $\mathbb{Z}^2$, therefore it is either trivial, $\cong \mathbb{Z}$ or $\cong \mathbb{Z}^2$. Since $U \cap N_{i_0-1}$ contains the nontrivial $N \neq \mathbb{Z}$, we must have $U \cap N_{i_0-1} \cong \mathbb{Z}^2$. Because a finite cover of $M_{N_{i_0}}$, hence also of M, fibers over the circle with fiber the torus, we have $|G : \pi_1(T) \rtimes \mathbb{Z}| < \infty$. If $U \cap N_{i_0-1}$ were not of finite index in U, then U would be of finite index in G, which contradicts the assumptions on U. Therefore we conclude that U is commensurable with $\pi_1(T)$, as desired.

Case 3: $N = N_0, N_1, N_2, ..., N_n = G$ are all finitely generated and there exists (at least) one value of the index i for which $|N_i : N_{i-1}| = \infty$

Analogously to Case 2 above, let i_0 be the largest value of the index i for which $|N_{i_0} : N_{i_0-1}| = \infty$. The argument in Case 2 above can be applied here as well, with $N = N_0$, $U = N_{i_0-1}$ for the application of Theorem 2.1, to reach the same conclusion as in Case 2. □

3. Correction to the proof of the Main Theorem - Theorem 5.3

Below is a restatement of the Main Theorem along with its correct proof, which addresses the point that the prime summands M_i may, in principle, have nonempty boundaries.

Theorem 5.3. *Let M be a compact 3-manifold with empty or toroidal boundary. If $G = \pi_1(M)$ contains a finitely generated subgroup U of infinite index in G which contains a nontrivial, subnormal subgroup N of G, then: (a) M is irreducible, (b) if further:*

(1) *N has a subnormal series of length n in which $n-1$ terms are assumed to be finitely generated, and*

(2) *either all inclusions* $N_i \hookrightarrow N_{i+1}$, *for* $i > 0$, *are of finite index, or there exist (at least two) indices* i_0 *and* i_1, $i_0 \neq i_1$, $i_0, i_1 > 1$, *such that the inclusions* $N_i \hookrightarrow N_{i+1}$ *are of infinite index for* $i = i_0, i_1$, *or* $N = N_0$ *is finitely generated and there exists (at least) one value of the index* i *for which the inclusion* $N_i \hookrightarrow N_{i+1}$ *is of infinite index, and*
(3) N *intersects nontrivially the fundamental groups of the splitting tori of some decomposition* $\mathfrak{D}$ *of* M *into geometric pieces, and*
(4) *the intersections of* N *with the fundamental groups of the geometric pieces are not isomorphic to* $\mathbb{Z}$,

then, M *has a finite cover which is a bundle over* $\mathbb{S}$ *with fiber a compact surface* F *such that* $\pi_1(F)$ *and* U *are commensurable.*

Proof. First we prove (a): By Theorem 1 in [2] $M \cong M_1 \sharp M_2 \sharp ... \sharp M_p$, where each M_i is a prime manifold. Then, we have $G = G_1 * G_2 * ... * G_p$, where $G_i = \pi_1(M_i)$. By Theorem 1.5 in [1], we must have $G_i = 1$ for $i \geq 2$, after possibly reindexing the terms. Now, each M_i, for $i \geq 2$, is an orientable, prime, compact, simply connected 3-manifold, whose boundary is empty or toroidal. Since $\pi_1(M_i) = \{1\}$, we see that M_i is irreducible as the only prime, orientable, 3-manifold, which is not irreducible is $\mathbb{S}^2 \times \mathbb{S}$, and also that M_i is geometric by Theorem 5.1. Therefore the interior of M_i, $int(M_i)$, is the finite volume quotient of one of the eight geometries by $\pi_1(int(M_i))$. Note that even if $\partial M_i \neq \emptyset$, $\pi_1(int(M_i)) \cong \pi_1(M_i)$ since $int(M_i)$ retracts onto the homeomorphic to M_i submanifold of M_i obtained by removing the interior of a collar neighborhood of ∂M_i. Thus we conclude that $int(M_i)$ is itself a geometry of finite volume, hence $int(M_i) = \mathbb{S}^3$. Since $\mathbb{S}^3$ is compact $cl(int(M_i)) = int(M_i)$. Now, because $\partial M_i \subset cl(int(M_i))$, we see that $\partial M_i = \emptyset$ and $M_i = \mathbb{S}^3$ for all $i \geq 2$, and we further conclude that M itself is a prime manifold. Therefore M must be irreducible, for if it were not, then $M \cong \mathbb{S}^2 \times \mathbb{S}$, hence $G = \mathbb{Z}$, $U = \{1\}$, contradicting the hypothesis of the theorem. To prove (b), we make an inductive argument to prove that every connected submanifold $M' \subseteq M$ which is a union of $X_i \in \mathfrak{D}$ has property (A') with respect to its decomposition $\mathfrak{D}'$ into geometric pieces inherited from $\mathfrak{D}$. We proceed by induction on the number m of geometric pieces X_i in M'. It is clearly true that if $M' = X_i$ for some i, then M' has property (A') by Theorem 3.9. Suppose that all submanifolds which are a union of at most $m-1$ geometric pieces from $\mathfrak{D}$ have property (A') with respect to their geometric decompositions along incompressible tori from $\mathfrak{D}$, and suppose that M' is a union of m geometric pieces from $\mathfrak{D}$ so that $\mathfrak{D}' = (X_{i_1}, ..., X_{i_m}; \mathcal{T}_{i_1}, ..., \mathcal{T}_{i_s})$. If we cut M' along all tori in $\mathfrak{D}'$ which are in X_{i_m}, M' will be decomposed into connected submanifolds $N_1, ..., N_w, X_{i_m}$, in such a way that each of the N_i has a geometric decomposition $\mathfrak{D}_i$ which consists of at most $m-1$ pieces from $\mathfrak{D}$, and tori also from $\mathfrak{D}$. Thus, it follows that each of the submanifolds $N_i, ..., N_w, X_{i_m}$ with its geometric decomposition inherited from $\mathfrak{D}$ has property (A'). Since M' can be obtained from the pieces $N_1, ..., N_w, X_{i_m}$ by performing a finite number of torus sums of the form $M_1 \cup_{\mathcal{T}} M_2$ or $M_1 \cup_{\mathcal{T}}$ for $M_i \in \{N_1, ..., N_w, X_{i_m}\}$ and $\mathcal{T} \in \mathfrak{D}$, M' also has property (A') with respect to its geometric decomposition along tori from $\mathfrak{D}$ by Proposition 5.2. By induction, M also has property (A') hence it fibers in the required way. □

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