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# ERRATUM TO: A FIBERING THEOREM FOR 3-MANIFOLDS

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Abstract. Erratum to the paper titled *A fibering theorem for 3-manifolds*, which appeared in print in the journal of Groups, Complexity, Cryptology Volume 13 Issue 2, published Nov. 11, 2021.

## 1. Overview

Much to my disappointment, I noticed few errors in my article since its publication, as well as a rather significant gap in [4]. Before I go into further detail, I will hasten to say that while the statement of Part(a) of my Main Theorem remain valid, Part (b) is significantly impacted by what appears to be an oversight in Theorem 2.10 in [4], which I myself failed to examine closely, to the extent that I now must state Part (b) as a result *contingent* on the successful proof of a theorem analogous to Theorem 2.9 in [4] for compact manifolds with boundary. The proof of Part (a) of the Main Theorem, however, also requires a correction - a matter which I address in Section 4 below.

From the statements on lines 6 and 7 on page 31 in [4], one may reasonably infer that the hypothesis of $M$ being closed in Theorem 2.9 can be relaxed to $M$ being compact only, and that the proof of Theorem 2.9 works with the obvious changes. Unfortunately, this does not appear to be the case, as the statement on lines 3 and 4 on page 31 is no longer true for the stated reasons: the cover $\widetilde{M}$ of $M''$ is now a manifold with boundary, as $S'$ is itself a surface with boundary. Hence, in this case, one cannot immediately conclude that $M' = M''$. Thus, Theorem 2.9 in [4] has only been proved for closed manifolds without the obvious generalization suggested by the statement labeled as Theorem 2.10 in [4].

Theorems 4.8 and 4.9 in my article, being reliant on the proof of Theorem 2.9 in [4] extending to compact manifolds with boundary, *are thus not proven results.*

I have also discovered an error in the proof of Case 2 of my Theorem 4.6, and certain minor errors and misprints, such as, for example, the word "nontrivial" missing from the hypothesis of Proposition 3.1, and the symbol $\mathbb{Z}$ having been incorrectly used to stand for any cyclic group in Lemma 3.3.

Rather than submit a complete retraction, I have decided to submit a corrigendum, since the results in Sections 2 and 3 of my article remain correct, and thus my theorems properly generalize all of the results proven in [4].

*Key words and phrases:* 3-manifolds, fiber bundles, Bass-Serre trees.

Finally, a few extra lines are needed to fix an error in Proposition 3.5 and a gap in the proof of Theorem 3.6, both due to the conclusion of Theorem 3 in [2] asserting the desired fibration result for the Poincare associate of $M$, rather than $M$ itself. Moon never explicitly states an irreduciliblity assumption on $M$, although he appears to have implicitly made such an assumption remarking on it in the first pages of his paper. Without it, his results need to be stated in terms of $\hat{M}$.

## 2. Corrections of Proposition 3.5 and Theorem 3.6

In Proposition 3.5, I cited Theorem 3 in [2], stated as Theorem 1.2 in my original article for ease of reference, to claim that, under the stated hypothesis, a 3-manifold $M$ fibers over the circle. This theorem, however, shows that the Poincare associate, $\hat{M}$, rather than $M$ itself, fibers in the desired way under the stated hypothesis. All that is needed to make Proposition 3.5 correct is the trivial observation in its proof that the Poincare associate of a finite cover is a finite cover of the Poincare associate, and change the statement of its conclusion so that $\hat{M}$ is asserted to fibers over the circle, rather than $M$. Now, as a consequence, the proof of Case 2 of Theorem 3.6 needs to take into account the case that when a Seifert fibered space $Y$ has a boundary, one might conceivably have $Y \neq \hat{Y}$. This, however, turns out not to be the case, since every compact, connected Seifert fibered space with a nonempty boundary is $P^2$-irreducible, see for example Lemma 2.1.4 in Matthew Brin's notes on Seifert fibered spaces, arXiv: 0711.1346, hence no boundary 2-spheres can exist, and $\hat{Y} = Y$.

## 3. Correction of Theorem 4.6

Here again I have made the mistake of not taking into account the matter that if there are 2-sphere components in $\partial M$, then the result is true for $\hat{M}$. In Case 2 of the proof of Theorem 4.6, I made the mistake of assuming part of the claim I am proving - namely, I have incorrectly assumed that the surface $T$, which is referenced on line 3 from the bottom on page 11 of the article, is the torus. Proving the theorem was possible under a somewhat more restrictive hypothesis. Below is a restatement of Theorem 4.6 and its proof under the revised hypothesis. This change in Theorem 4.6 requires that the hypotheses and proofs of all the subsequent results which rely on Theorem 4.6, including the definitions of property $(A)$ and property $(A')$, be changed in the obvious way, pending a successful proof of Theorem 2.9 in [4] for compact manifolds.

**Theorem 4.6.** *Let $M$ be a compact 3-manifold with $\pi_1(M) = G$, and suppose that $M$ splits along an incompressible torus $\mathcal{T}$, $M = X_1 \cup_{\mathcal{T}} X_2$, or $M = X_1 \cup_{\mathcal{T}}$. Suppose that:*

(1) *$G$ contains a nontrivial, subnormal subgroup $N = N_0 \lhd ... \lhd N_{n-1} \lhd N_n = G$ such that $N \neq \mathbb{Z}$ and,*
(2) *at least $n$ terms in the subnormal series $N = N_0 \lhd ... \lhd N_{n-1} \lhd N_n = G$ are finitely generated and,*
(3) *either all inclusions $N_i \hookrightarrow N_{i+1}$, for $i > 0$, are of finite index, or there exist indices $i_0$ and $i_1$, $i_0 \neq i_1$, $i_0, i_1 > 1$, such that the inclusions $N_i \hookrightarrow N_{i+1}$ are of infinite index for $i = i_0, i_1$, or $N = N_0$ is finitely generated and there exists a value of the index $i$ for which the inclusion $N_i \hookrightarrow N_{i+1}$ is of infinite index and,*
(4) *$G$ contains a finitely generated subgroup $U$ of infinite index in $G$ such that $N < U$.*

*If the graph of groups $\mathcal{U}$ corresponding to $U$ has infinite diameter, then $\widehat{M}$ is finitely covered by a torus bundle over $\mathbb{S}^1$ with fiber $T$, and $U$ and $\pi_1(T)$ are commensurable.*

*Proof.* ...

*Case 2*: $|N_i : N_{i-1}| = \infty$ for $i = i_1, i_2 > 1$, where $i_1 \neq i_2$

Let $i_0$ be the largest integer index for which $|N_{i_0} : N_{i_0-1}| = \infty$. In this case, we consider the finite cover $M_{N_{i_0}}$ whose fundamental group is $N_{i_0}$. Since $N_{i_0-1}$ is assumed to be finitely generated, it is also finitely presented by Theorem 2.1 in [5]. Now, by Theorem 1.2 either $\widehat{M_{N_{i_0}}}$ itself or, in the case when $\widehat{M_{N_{i_0}}}$ is a union of two twisted $I$-bundles, a finite cover of $\widehat{M_{N_{i_0}}}$ fibers over $\mathbb{S}^1$ with fiber a compact surface $F$ and, further, $N_{i_0-1}$ is a subgroup of finite index in $\pi_1(F)$. Since there are at least two distinct values of the index $i$, for which $|N_i : N_{i-1}| = \infty$, there is a value of the index $1 < k < i_0$, for which $|N_k : N_{k-1}| = \infty$. Consider now the finite cover $F_{N_{i_0-1}}$ of $F$, whose fundamental group is $N_{i_0-1}$. Since $N_{k-1}$ is assumed to be finitely generated, applying Theorem 2.1 with $N = N_0$ and $U = N_{k-1}$, we conclude that $F_{N_{i_0-1}}$ is the torus or the Klein bottle. Hence, $F$ itself is the torus or the Klein bottle. Because $M_{N_{i_0}}$, being a finite cover of the orientable manifold $M$, is itself an orientable 3-manifold, we see that the Poincare associate $\widehat{M_{N_{i_0}}}$ is orientable. However, all four Klein bottle bundles over $\mathbb{S}$ are non-orientable 3-manifolds and we conclude that $F$ is the torus $T$ and that $N_{i_0-1}$ is a subgroup of finite index in $\pi_1(T)$, which is a group isomorphic to $\mathbb{Z}^2$.

Finally, we show that $U$ is commensurable with $\pi_1(T)$. Consider $U \cap N_{i_0-1}$; this group is a subgroup of $\mathbb{Z}^2$, therefore it is either trivial, $\cong \mathbb{Z}$ or $\cong \mathbb{Z}^2$. Since $U \cap N_{i_0-1}$ contains the nontrivial $N \neq \mathbb{Z}$, we must have $U \cap N_{i_0-1} \cong \mathbb{Z}^2$. Because a finite cover of $\widehat{M_{N_{i_0}}}$, hence also of $\widehat{M}$, fibers over the circle with fiber the torus, we have $|G : \pi_1(T) \rtimes \mathbb{Z}| < \infty$. If $U \cap N_{i_0-1}$ were not of finite index in $U$, then $U$ would be of finite index in $G$, which contradicts the assumptions on $U$. Therefore we conclude that $U$ is commensurable with $\pi_1(T)$, as desired.

*Case 3*: $N = N_0, N_1, N_2, ..., N_n = G$ are all finitely generated and there exists (at least) one value of the index $i$ for which $|N_i : N_{i-1}| = \infty$

Analogously to Case 2 above, let $i_0$ be the largest value of the index $i$ for which $|N_{i_0} : N_{i_0-1}| = \infty$. The argument in Case 2 above can be applied here as well, with $N = N_0$, $U = N_{i_0-1}$ for the application of Theorem 2.1, to reach the same conclusion as in Case 2. □

## 4. Correction to the proof of Part(a) of the Main Theorem - Theorem 5.3

Below is a restatement of Part (a) of the Main Theorem along with its correct proof, which addresses the point that the prime summands $M_i$ may, in principle, have nonempty boundaries.

**Theorem 5.3, Part(a).** *Let $M$ be a compact 3-manifold with empty of toroidal boundary. If $G = \pi_1(M)$ contains a finitely generated subgroup $U$ of infinite index in $G$ which contains a nontrivial, subnormal subgroup $N$ of $G$, then $M$ is irreducible.*

*Proof.* By Theorem 1 in [3] $M \cong M_1 \sharp M_2 \sharp ... \sharp M_p$, where each $M_i$ is a prime manifold. Then, we have $G = G_1 * G_2 * ... * G_p$, where $G_i = \pi_1(M_i)$. By Theorem 1.5 in [1], we must have $G_i = 1$ for $i \geq 2$, after possibly reindexing the terms. Now, each $M_i$, for $i \geq 2$, is an orientable, prime, compact, simply connected 3-manifold, whose boundary is empty or toroidal. Since $\pi_1(M_i) = \{1\}$, we see that $M_i$ is irreducible as the only prime, orientable, 3-manifold, which is not irreducible is $\mathbb{S}^2 \times \mathbb{S}$, and also that $M_i$ is geometric by Theorem 5.1. Therefore the interior of $M_i$, $int(M_i)$, is the finite volume quotient of one of the eight geometries by $\pi_1(int(M_i))$. Note that even if $\partial M_i \neq \emptyset$, $\pi_1(int(M_i)) \cong \pi_1(M_i)$ since $int(M_i)$ retracts onto the homeomorphic to $M_i$ submanifold of $M_i$ obtained by removing the interior of a collar neighborhood of $\partial M_i$. Thus we conclude that $int(M_i)$ is itself a geometry of finite volume, hence $int(M_i) = \mathbb{S}^3$. Since $\mathbb{S}^3$ is compact $cl(int(M_i)) = int(M_i)$. Now, because $\partial M_i \subset cl(int(M_i))$, we see that $\partial M_i = \emptyset$ and $M_i = \mathbb{S}^3$ for all $i \geq 2$, and we further conclude that $M$ itself is a prime manifold. Therefore $M$ must be irreducible, for if it were not, then $M \cong \mathbb{S}^2 \times \mathbb{S}$, hence $G = \mathbb{Z}$, $U = \{1\}$, contradicting the hypothesis of the theorem. □

Again, I reiterate that the proof of Part (b) is not yet finished as it relies on a fibration result analogous to Theorem 2.9 in [4] for compact manifolds with boundary under the stated hypothesis on $\pi_1(M)$, as the inductive argument of the Main Theorem makes use of cutting along certain incompressible tori, thus even if $M$ were assumed closed, the geometric chunks obtained after an application of the Geometrization Theorem would have toroidal boundaries.

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