

A NOTE ON CO-HOPFIAN GROUPS AND RINGS

ANTHONY M. GAGLIONE AND DENNIS SPELLMAN

03 Elmwood Court, Arnold, MD 21012 U.S.A.
e-mail address: agaglione@aol.com

1134 Haverford Road, Apt. A, Crum Lynne, PA19022 U.S. A.
e-mail address: dennisspellman1@aol.com

ABSTRACT. Let p and n be positive integers. Assume additionally that $p \neq 3$ is a prime and that $n > 2$. Let R be a field of characteristic p . A very special consequence of a result of Bunina and Kunyavskii (2023, arXiv:2308.10076) is that $SL_n(R)$ is co-Hopfian as a group if and only if R is co-Hopfian as a ring. In this paper, we prove that if k is the algebraic closure of the 2 element field, then $SL_2(k)$ is a co-Hopfian group. Since this k is trivially seen to be co-Hopfian as a ring our result somewhat extends that of Bunina and Kunyavskii. We apply our result to prove that the class of groups satisfying Turner's Retract Theorem (called Turner groups here) is not closed under elementary equivalence thereby answering a question posed by the authors in (2017, Comm. Algebra).

In memory of Ben Fine

INTRODUCTION

According to the account in [6], in 1932 Heinz Hopf posed the question of whether or not a finitely generated group could be isomorphic to a proper homomorphic image. Hence a group is *Hopfian* if it is not isomorphic to a proper homomorphic image. Of course every finite group is Hopfian; so this finiteness property is of interest for infinite groups. As the reader is no doubt well aware efforts to answer this question were successful and there is a vast literature treating both Hopfian and non-Hopfian groups. Clearly the Hopf property can be stated in a universal algebraic context and it makes sense to speak, for example, of Hopfian and non-Hopfian rings. For us a ring is an associative \mathbb{Z} -algebra with multiplicative identity $1 \neq 0$. Subrings are required to contain 1 and homomorphisms are required to preserve 1. More recently the dual property has generated some interest. An algebra of some fixed type (for us group or ring) is *co-Hopfian* provided it is not isomorphic to a proper subalgebra. Of course every finite algebra is co-Hopfian; so, this finiteness property is of interest for infinite algebras. We note that equivalent formulations of both Hopficity and co-Hopficity can be stated in terms of endomorphisms. Namely, an algebra is Hopfian if and

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only if every surjective endomorphism is an automorphism and an algebra is co-Hopfian if and only if every injective endomorphism is an automorphism.

Enough pandering! We fix notation. If p is a prime $\overline{\mathbb{F}}_p$ shall be the algebraic closure of the p element field. For the remainder of this paper k shall be $\overline{\mathbb{F}}_2$ and G , standing alone, shall always be the group $SL_2(k)$.

1. EXAMPLES

We first note that if p is a prime then $\overline{\mathbb{F}}_p$ is co-Hopfian as a ring. Of course, since fields are simple, every endomorphism is injective. Assume to deduce a contradiction that the endomorphism $\varphi : \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p$ is not surjective. Suppose the image of φ omits the element θ of $\overline{\mathbb{F}}_p$. Let $f = \text{Irr}(\mathbb{F}_p, \theta)$ be the minimum polynomial of θ over the p element field \mathbb{F}_p . Suppose f has degree n and that the n roots of f in $\overline{\mathbb{F}}_p$ are $\theta = \theta_1, \theta_2, \dots, \theta_n$. Since φ is injective it must permute the θ_i , so there must be an i , $1 \leq i \leq n$, such that $\varphi(\theta_i) = \theta$ - contrary to hypothesis. The contradiction shows every endomorphism is surjective; hence, as claimed, $\overline{\mathbb{F}}_p$ is co-Hopfian.

A similar argument shows that the multiplicative group $\overrightarrow{\mathbb{F}}_p = \overline{\mathbb{F}}_p \setminus \{0\}$ of $\overline{\mathbb{F}}_p$ is co-Hopfian. To see that observe that $\overrightarrow{\mathbb{F}}_p$ is locally cyclic being the direct union of the family $\mathbb{F}_{p^n}^*$ of subgroups cyclic of order $p^n - 1$ as n varies over the positive integers. Let ϕ be the Euler totient and, with $N = \phi(p^n - 1)$, let $\{\theta_1, \dots, \theta_N\}$ be the N elements of order $p^n - 1$ in $\mathbb{F}_{p^n}^*$. These are precisely the N roots of the cyclotomic polynomial of degree N over \mathbb{F}_p . Every injective homomorphism of $\overrightarrow{\mathbb{F}}_p$ must permute these. Since (taken over all n) these generate $\overrightarrow{\mathbb{F}}_p$, every injective homomorphism of $\overrightarrow{\mathbb{F}}_p$ is surjective and as claimed $\overrightarrow{\mathbb{F}}_p$ is co-Hopfian as a group.

Ol'shanskii has constructed examples of groups Γ with the following properties.

- (1) Γ is infinite
- (2) Γ is nonabelian
- (3) Γ is 2-generator
- (4) Every proper subgroup of Γ is cyclic.

Let us call a group satisfying 1,2,3,4 above an *Ol'shanskii group*. Clearly no Ol'shanskii group can be isomorphic to a proper subgroup; so every Ol'shanskii group is co-Hopfian.

In the course of proving a rigidity result Bunina and Kunyavskii established in [2] a preliminary proposition on Chevalley group $G(\Phi, R)$ over local rings R . Here Φ is a reduced irreducible root system of rank at least 2.

Proposition 1.1 ([2]). *With the conventions and notation above and with the hypotheses that for $\Phi = A_2, B_l, C_l, \mathbb{F}_4$, 2 is a unit in R and for $\Phi = G_2$, 3 is a unit in R , it is the case that $G(\Phi, R)$ is co-Hopfian as a group if and only if R is co-Hopfian as a ring.*

Recalling that $k = \overline{\mathbb{F}}_2$, we thank the referee for pointing out to us that, for $n > 2$, it follows from Proposition 1 that $SL_n(k)$ is co-Hopfian.

In the next section, we shall prove that $G = SL_2(k)$ is co-Hopfian.

2. THE CO-HOPFICITY OF G

We first observe that, since k is algebraically closed, the Frobenius map $\sigma: k \rightarrow k$, $x \mapsto x^2$ is surjective. Put another way, every element of k has a unique square root.

Notation 2.1. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(k)$, then $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ shall be its determinant $ad + bc$.

We next observe that if $X \in GL_2(k)$, then conjugation by $Y = \frac{1}{\sqrt{\det(X)}} \cdot X$ has the same effect as conjugation by X ; moreover, Y lies in G .

It follows from this that two elements of G conjugate in $GL_2(k)$ must already be conjugate in G .

Henceforth we denote the multiplicative and additive groups of k by k^* and k^+ respectively.

Every element of $GL_2(k)$ is conjugate to one of the following Jordan canonical forms, unique up to the order of the blocks:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \lambda \in k^*, \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \lambda, \mu \in (k^*)^2 \setminus \{(1, 1)\}.$$

It follows that every element of G is conjugate in G to one of the following Jordan canonical forms, unique up to the order of the blocks:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \lambda \in k^*, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda \in k^* \setminus \{1\}.$$

Remark 2.2. (1) $\lambda = \lambda^{-1}$ if and only if $\lambda^2 = 1$ if and only if $\lambda = 1$.

(2) Since each of taking transposes and taking inverses is an automorphism of G , their composition, in either order

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

is an automorphism of G .

(3) The element $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of G has order 2 and the inner automorphism determined by this element is the inverse transpose.

(4) Thus, if $\lambda \in k^* \setminus \{1\}$, then $\lambda \neq \lambda^{-1}$ and the matrices $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ and $\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}$ are conjugate in G .

Note that the element $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of G has order 2 while, if $\lambda \in k^* \setminus \{1\}$, then the order of $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in G$ is the order of λ in k^* and that, being a divisor of $2^n - 1$ for some n , is odd.

Note also that, for all $\lambda \in k^* \setminus \{1\}$, $\lambda + \lambda^{-1} = 0$ if and only if $\lambda = \lambda^{-1}$ if and only if $\lambda = 1$. It follows that every element of G either has odd order or has order 2 and, moreover, the elements of order 2 are precisely the nontrivial elements of trace 0.

Definition 2.3. A group Γ which satisfies the universal sentence

$$\forall x, y, z ((y \neq 1) \wedge (xy = yx) \wedge (yz = zy)) \rightarrow (xz = zx))$$

is *commutative transitive*, briefly *CT*.

Proposition 2.4 ([5]). *Let Γ be a group. The following three statements are pairwise equivalent.*

- (1) Γ is CT
- (2) For each $g \in \Gamma \setminus \{1\}$, its centralizer is abelian.
- (3) If M_1 and M_2 are maximal abelian subgroups in Γ , then $M_1 \cap M_2 = \{1\}$ unless $M_1 = M_2$.

Remark 2.5. In any CT group, the maximal abelian subgroups are the centralizers of nontrivial elements.

Theorem 2.6 ([9]). *Let Γ be a finite nonsolvable group. Then Γ is CT if and only if it is isomorphic to $SL_2(\mathbb{F}_{2^n})$ for some integer $n \geq 2$.*

Since G is the direct union $\varinjlim (SL_2(\mathbb{F}_{2^n}))$ and universal sentences are preserved in direct unions, we have the following immediate corollary -

Corollary 2.7. *G is CT.*

We make explicit some terminology we shall use going forward.

Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in G$. We call X *diagonal* if it has the form $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$ ($x \in k^*$); *off diagonal* if it has the form $\begin{bmatrix} 0 & y \\ y^{-1} & 0 \end{bmatrix}$ ($y \in k^*$); *upper triangular* if it has the form $\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix}$ ($x \in k^*$); *upper unitriangular* if it has the form $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$; *lower triangular* if it has the form $\begin{bmatrix} x & 0 \\ z & x^{-1} \end{bmatrix}$ ($x \in k^*$); *lower unitriangular* if it has the form $\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$.

We write $\Delta, \Delta', U, UT, L$ and LT for the sets of diagonal, off diagonal, upper triangular, upper unitriangular, lower triangular and lower unitriangular matrices respectively.

Now let $\begin{bmatrix} s & t \\ u & v \end{bmatrix} \in G$ and $\lambda \in k^*$. We explicitly compute the following two conjugations.

$$\begin{bmatrix} s & t \\ u & v \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} v & t \\ u & s \end{bmatrix} = \begin{bmatrix} \lambda sv + \lambda^{-1} tu & (\lambda + \lambda^{-1})st \\ (\lambda + \lambda^{-1})uv & \lambda^{-1}sv + \lambda tu \end{bmatrix} \quad (2.1)$$

$$\begin{bmatrix} s & t \\ u & v \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v & t \\ u & s \end{bmatrix} = \begin{bmatrix} 1 + \lambda su & \lambda s^2 \\ \lambda u^2 & 1 + \lambda su \end{bmatrix} \quad (2.2)$$

Proposition 2.8. *Let $\lambda \in k^* \setminus \{1\}$ and $g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$. Then $\Delta = C_G(g)$.*

Proof. We must find $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$ which conjugates g to g . With an eye towards also determining $N_G(\Delta)$, we use the conjugation computation in Eq. 2.1 to find more generally, the $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$ which conjugate g into Δ . Since $\lambda \neq 1$, we have $\lambda + \lambda^{-1} \neq 0$. Then $st = uv = 0$. Since $sv + tu = 1$ there are two possibilities, namely:

$$s = v = 0 \text{ or } t = u = 0.$$

If $s = v = 0$, then $\begin{bmatrix} s & t \\ u & v \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} v & t \\ u & s \end{bmatrix} = \begin{bmatrix} 0 & t \\ t^{-1} & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} 0 & t \\ t^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \neq \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ since $\lambda \neq 1$.

So $t = u = 0$ and $\begin{bmatrix} s & t \\ u & v \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} v & t \\ u & s \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} s^{-1} & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$ □

Corollary 2.9. *If $g \in G \setminus \{1\}$ has odd order, then $C_G(g)$ is isomorphic to k^* .*

Proposition 2.10. *$N_G(\Delta)$ is generated by Δ' . It is metabelian and is a semidirect product of Δ by the cycle of order 2 generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. By the proof of Proposition 3, $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$ conjugates Δ into itself if and only if it lies in either Δ or Δ' . From the equivalent equations

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (\lambda \in k^*)$$

we see that $N_G(\Delta)$ is generated by Δ' and that it is the product of $\Delta \trianglelefteq N_G(\Delta)$ and $\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$. Since these intersect in the identity, $N_G(\Delta)$ is their semidirect product. □

Corollary 2.11. *If $g \neq 1$ has odd order and $M = C_G(g)$, then $N_G(M)$ is metabelian and is a semidirect product of k^* by a cycle of order 2.*

Proposition 2.12. *Let $\lambda \in k^*$ and $g = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. Then $UT = C_G(g)$.*

Proof. We must find $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$ which conjugates g to g . With an eye towards also determining $N_G(UT)$, we use the conjugation computation in Eq. 2.2 to find, more generally, the $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$ which conjugate g into UT . From that computation we see that this entails $\lambda u^2 = 0$ and since $\lambda \neq 0$ we see that $u = 0$; so

$$\begin{bmatrix} s & t \\ u & v \end{bmatrix} = \begin{bmatrix} s & t \\ 0 & s^{-1} \end{bmatrix} \in U.$$

Again, from Eq. 2.2, we see that in order that $\begin{bmatrix} s & t \\ u & v \end{bmatrix} = \begin{bmatrix} s & t \\ 0 & s^{-1} \end{bmatrix}$ to conjugate g into g we must have $\lambda s^2 = \lambda$ and since $\lambda \neq 0$, $s^2 = 1$ and hence $s = 1$. Thus, $C_G(g)$ consists of the $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and so coincides with UT . □

Corollary 2.13. *By taking the inverse transpose automorphism we see that, if $\lambda \in k^*$, and $g = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$, then $C_G(g) = LT$.*

Corollary 2.14. *If $g \in G \setminus \{1\}$ has order 2, then $C_G(g)$ is isomorphic to k^+ .*

Proposition 2.15. *$N_G(UT) = U$. It is metabelian and a semidirect product of UT by Δ .*

Proof. From Proposition 5, we get $N_G(UT) = U$. Given $\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix} \in U$ ($x \in k^*$) we have $\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix} = \begin{bmatrix} 1 & xy \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$; so, U is the product of the subgroups $UT \trianglelefteq N_G(UT) = U$ and Δ . Moreover, these subgroups intersect in the identity; so U is their semidirect product. \square

Remark 2.16. The elements of order 2 in U are precisely the nontrivial elements of UT since $x + x^{-1} = 0$ implies $x = 1$.

Corollary 2.17. *By taking the inverse transpose automorphism we see that $N_G(LT) = L$. It is metabelian and a semidirect product of LT by Δ .*

Corollary 2.18. *If g has order 2 and $M = C_G(g)$, then $N_G(M)$ is metabelian and is a semidirect product of k^+ by k^* .*

Proposition 2.19. *Let $(\lambda_1, \lambda_2) \in (k^*)^2$. Then $\begin{bmatrix} 1 & \lambda_1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \lambda_2 & 1 \end{bmatrix}$ do not commute.*

Proof. The centralizers of $\begin{bmatrix} 1 & \lambda_1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \lambda_2 & 1 \end{bmatrix}$ are UT and LT respectively and these have trivial intersection. \square

Every element of order 2 in G is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which lies in the subgroup $SL_2(\mathbb{F}_4)$; $SL_2(\mathbb{F}_4)$ is a simple group of order 60. Every simple group of order 60 is isomorphic to the alternating group A_5 (see [3]). Since A_5 is generated by 3-cycles, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the product of elements of order 3. Thus G is generated by its elements of odd order. Every element of odd order is conjugate to an element of Δ . Given $\lambda \in k^*$ we have

$$\begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix};$$

so, every element of odd order is the product of two elements of order 2 and G is generated by its elements of order 2. These are conjugates of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and, by Eq. 2.2, have the form

$$\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix}.$$

These depend on s and u only; so, if $s \neq 0$, we can take the conjugating matrix to be the lower triangular matrix $\begin{bmatrix} s & 0 \\ u & s^{-1} \end{bmatrix}$ while if $s = 0$ then $\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u^2 & 1 \end{bmatrix}$ is itself lower triangular.

(Note that the above is an arbitrary element of LT as the Frobenius map $k \rightarrow k$, $x \mapsto x^2$ is an automorphism.) It follows that G is generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and L . Now $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in L$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ conjugates $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; so G is generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and L . It is therefore generated by the larger set Δ' and L and hence generated by the subgroups $N_G(\Delta)$ and L .

Now suppose $\varphi: G \rightarrow G$ is an injective endomorphism. Let $\theta \in k$ be a primitive cube root of unity. Let g be the order 3 element $\begin{bmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{bmatrix}$.

Then $\varphi(g)$ has order 3 so is a conjugate to $\begin{bmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{bmatrix}$.

Let the inner automorphism α conjugate $\varphi(g)$ to g . Then $\alpha\varphi$ is also an injective endomorphism. Since $\alpha\varphi$ fixes g it must map $\Delta = C_G(g)$ into Δ and so restricts to an injective endomorphism of Δ . Now Δ is isomorphic to k^* which, by the example $\overrightarrow{\mathbb{F}_p}$, of Section 2 with $p = 2$ is co-Hopfian. Hence, $\alpha\varphi$ restricts to an automorphism of Δ . Then $\alpha\varphi$ maps $N_G(\Delta)$ into $N_G(\Delta)$. Now $\alpha\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ maps to an element of order 2 so must lie in Δ' . Say $\alpha\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix}$. Since $\Delta \leq \text{Im}(\alpha\varphi)$, $\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ lies in $\text{Im}(\alpha\varphi)$. Now $N_G(\Delta)$ is generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and Δ ; so $\alpha\varphi$ restricts to an automorphism of $N_G(\Delta)$.

What does $\alpha\varphi$ do on L ? L is generated by Δ and elements of order 2 which are conjugated by Δ into commuting elements of order 2. What elements of order 2 are conjugated by Δ into elements which commute with the original? As we have seen before an arbitrary element of order 2 has the form

$$\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix}$$

with $s \neq 0$ or $u \neq 0$. Let $\lambda \in k^* \setminus \{1\}$. Computing

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}$$

we get

$$\begin{bmatrix} 1 + su & \lambda^2 s^2 \\ \lambda^{-2} u^2 & 1 + su \end{bmatrix}.$$

When does this commute with $\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix}$?

$$\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix} \begin{bmatrix} 1 + su & \lambda^2 s^2 \\ \lambda^{-2} u^2 & 1 + su \end{bmatrix} = \begin{bmatrix} (1 + su)^2 + \lambda^{-2} s^2 u^2 & * \\ * & * \end{bmatrix}$$

$$\begin{bmatrix} 1 + su & \lambda^2 s^2 \\ \lambda^{-2} u^2 & 1 + su \end{bmatrix} \begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix} = \begin{bmatrix} (1 + su)^2 + \lambda^2 s^2 u^2 & * \\ * & * \end{bmatrix}$$

so $\lambda^2 s^2 u^2 = \lambda^{-2} s^2 u^2$ and $\lambda su = \lambda^{-1} su$ and $(\lambda + \lambda^{-1})su = 0$. Since $\lambda \neq 1$, $\lambda + \lambda^{-1} \neq 0$. Therefore, either $s = 0$ or $u = 0$. It follows that

$$\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u^2 & 1 \end{bmatrix} \in LT$$

or

$$\begin{bmatrix} 1 + su & s^2 \\ u^2 & 1 + su \end{bmatrix} = \begin{bmatrix} 1 & s^2 \\ 0 & 1 \end{bmatrix} \in UT.$$

By Proposition 7 we cannot have nontrivial instances of both LT and UT in the image of $\alpha\varphi$ on L . Now let β be the identity automorphism if the image of $\alpha\varphi$ on L contains nontrivial elements of LT and be the inverse transpose automorphism (conjugation by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) if the image of $\alpha\varphi$ on L contains nontrivial elements of UT .

The map φ will be an automorphism if and only if $\beta\alpha\varphi$ is. The map $\beta\alpha\varphi$ leaves $N_G(\Delta)$ alone and maps L into L .

$$\beta\alpha\varphi \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

for some $\lambda \in k^*$. Let $z \in k^*$ be arbitrary. Then

$$\begin{bmatrix} \sqrt{\lambda z^{-1}} & 0 \\ 0 & \sqrt{\lambda^{-1}z} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\lambda^{-1}z} & 0 \\ 0 & \sqrt{\lambda z^{-1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$$

so an arbitrary element of LT lies in the image of $\beta\alpha\varphi$ on L .

Since L is generated by Δ and LT , $\beta\alpha\varphi$ restricts to an automorphism of L . Since $N_G(\Delta)$ and L , $\beta\alpha\varphi$ is an automorphism of G and thus φ is also.

We have proven:

Theorem 2.20. *G is co-Hopfian.*

3. AN APPLICATION TO TURNER GROUPS

Definition 3.1. An element g of a group Γ is a *test element* provided every endomorphism of Γ which fixes g is an automorphism.

Definition 3.2 ([7]). A hyperbolic group Γ is *stably hyperbolic* if for each endomorphism $\varphi: \Gamma \rightarrow \Gamma$ and each positive integer n , there is an integer $m \geq n$ such that $\varphi^m(\Gamma)$ is hyperbolic.

Remark 3.3. Finite groups and finitely generated free groups are stably hyperbolic.

In [7] O’Neil and Turner proved

Theorem 3.4. *In any stably hyperbolic group an element is a test element if and only if it lies in no proper retract.*

Remark 3.5. Clearly lying in no proper retract is a necessary condition to be a test element. So the real theorem is that in stably hyperbolic groups this condition is sufficient.

Definition 3.6 ([4]). A group G is a *Turner group* provided every element which is excluded by every proper retract is a test element.

It was shown in [4] that the class of Turner groups is not the model class of any set of first order sentences in a language appropriate for group theory. In that paper it was posed as an open question whether or not the class of Turner groups satisfies the weaker condition of closure under elementary equivalence. We shall see below that it follows from the co-Hopfity of G the class of Turner groups is not closed under elementary equivalence.

Proposition 3.7 ([4]). *Every co-Hopfian simple group is a Turner group.*

Proof. Let Γ be a co-Hopfian simple group. Since Γ is simple the only proper retract is the trivial group $\{1\}$. Let $g \in \Gamma \setminus \{1\}$ and let the endomorphism $\varphi: \Gamma \rightarrow \Gamma$ fix g . Then $g \notin \text{Ker}(\varphi)$ so $\text{Ker}(\varphi) \neq \Gamma$ and hence $\text{Ker}(\varphi) = \{1\}$. Therefore, φ is injective. By co-Hopfity, φ is an automorphism. \square

Corollary 3.8. *G is a Turner group.*

Proof. $G = SL_2(k) = PSL_2(k)$ is a co-Hopfian simple group. \square

Proposition 3.9 ([4]). *Let K be a field of characteristic 2 which is not co-Hopfian as a ring. Then $SL_2(K)$ is not a Turner group.*

Proof. Let K_0 be a proper subfield of K and let $\varphi: K \rightarrow K_0$ be an isomorphism. Then φ induces an endomorphism $\bar{\varphi}: SL_2(K) \rightarrow SL_2(K)$ via

$$\bar{\varphi} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{bmatrix}$$

$\bar{\varphi}$ fixes every element of $SL_2(\mathbb{F}_2)$ but $\bar{\varphi}$ is not an automorphism since, if $\theta \in K \setminus K_0$ then e.g. $\begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix}$ does not lie in the image of $\bar{\varphi}$. \square

Now let t be transcendental over $k = \overline{\mathbb{F}_2}$. Let K be the algebraic closure of the transcendental extension $k(t)$ of k . Since the theory of algebraically closed fields of characteristic 2 is complete (a consequence of [1, Chapter 9, Corollary 1.11]) the fields k and K are elementarily equivalent. By the Keisler-Shelah Ultrapower Theorem [8]; there is a nonempty index set I and an ultrafilter D on I such that the ultrapowers $*K = K^I/D$ and $*k = k^I/D$ are isomorphic. Then $*K$ is isomorphic to a proper subfield $*k$ and then by Proposition 9, $SL_2(*K)$ is not a Turner group.

Now we have isomorphisms

$$\begin{aligned} SL_2(*K) &\cong SL_2(K)^I/D \\ &\cong SL_2(k)^I/D \\ &= *SL_2(k) \end{aligned}$$

and the class of Turner groups is not closed under ultrapowers hence not closed under elementary equivalence.

Remark 3.10. The argument also shows that the classes of co-Hopfian rings and groups are not closed under elementary equivalence.

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