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ANDREWS-CURTIS GROUPS

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Abstract. For any group G and integer $k \geq 2$ the Andrews-Curtis transformations act as a permutation group, termed the Andrews-Curtis group $AC_k(G)$, on the subset $N_k(G) \subset G^k$ of all k-tuples that generate G as a normal subgroup (provided $N_k(G)$ is non-empty). The famous Andrews-Curtis Conjecture is that if G is free of rank k, then $AC_k(G)$ acts transitively on $N_k(G)$. The set $N_k(G)$ may have a rather complex structure, so it is easier to study the full Andrews-Curtis group $FAC(G)$ generated by AC-transformations on a much simpler set G^k. Our goal here is to investigate the natural epimorphism $\lambda\colon FAC_k(G) \to AC_k(G)$. We show that if G is non-elementary torsion-free hyperbolic, then $FAC_k(G)$ acts faithfully on every nontrivial orbit of G^k, hence $\lambda\colon FAC_k(G) \to AC_k(G)$ is an isomorphism.

In memory of Ben Fine

1. The Andrews-Curtis Conjecture

Andrews-Curtis groups were introduced in connection with the Andrews-Curtis Conjecture (ACC) proposed by James J. Andrews and Morton L. Curtis in 1965 [4]. According to this conjecture a presentation P which is balanced (the number of its relators equals the number of its generators) presents the trivial group if and only if P can be reduced to the standard presentation of the trivial group by Andrews-Curtis transformations (defined below) of its sequence of relators. In other words $\langle x_1, \ldots, x_k \mid u_1, \ldots, u_k\rangle$ presents the trivial group if and only if $(u_1, \ldots, u_k)$ is AC-equivalent to $(x_1, \ldots, x_k)$.

Many people believe that the ACC is false. Indeed searching for counterexamples is currently an active area of research. In 1985, Akbulut and Kirby [2] suggested a sequence of potential counterexamples to the ACC of rank 2:

$$AK(n) = \langle x, y|\ x^n = y^{n+1},\ xyx = yxy\rangle,\ n \geq 2.$$

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The presentations $AK(n)$ are balanced presentations of the trivial group, and it was conjectured that that the pair of relators $(x^n y^{-n-1}, xyxy^{-1}x^{-1}y^{-1})$ is not AC-equivalent to the pair of generators(x, y). It turned out later that the presentation $AK(2)$ is AC-trivializable (see [18, 19]), so $AK(2)$ is not a counterexample to the ACC. The question whether or not the presentations $AK(n)$ with $n > 2$ are trivializable is still open despite an ongoing effort by the research community. Currently, $AK(3)$ is the shortest (in the total length of relators) potential counterexample to the ACC. Indeed, it was proved in [14] that if $\langle x, y \mid u = 1, v = 1\rangle$ is a presentation of the trivial group with $|u| + |v| \leq 13$ then either $(u, v) \sim_{AC} (x, y)$ or $(u, v) \sim_{AC} (x^3y^{-4}, xyxy^{-1}x^{-1}y^{-1})$. See papers [17, 8, 18, 19, 22, 23] for more details and some particular results. It was shown in [7] that AC-trivializations could be complex and exponentially long, so brute-force search algorithms are not going to work easily. Recently, methods of Reinforcement Learning were used with success in search for AC-trivializations of several known balanced presentations of the trivial group, see [26]; as well as axiomatic theorem proving [16].

To construct a counterexample to the ACC one might study the group structure of AC-transformations in an arbitrary group G and apply this knowledge to the ACC. For example, if the presentation $AK(3)$ is not AC-trivializable in a group G, then it is not AC-trivializable in F_2, hence the ACC fails. Below we initiate a study of the group of AC-transformations of a non-elementary torsion-free hyperbolic group.

2. AC transformations

AC transformations can be defined for any group G. The following are the *elementary Andrews-Curtis transformations* (or AC-moves) on G^k, where $k \geq 2$ is a natural number and G^k is the direct power of k copies of G.

(R_{ij}) $(u_1, \dots, u_i, \dots, u_k) \longrightarrow (u_1, \dots, u_i u_j^{\pm 1}, \dots, u_k)$, $i \neq j$;

(L_{ij}) $(u_1, \dots, u_i, \dots, u_k) \longrightarrow (u_1, \dots, u_j^{\pm 1} u_i, \dots, u_k)$, $i \neq j$;

(I_i) $(u_1, \dots, u_i, \dots, u_k) \longrightarrow (u_1, \dots, u_i^{-1}, \dots, u_k)$;

$(C_{i,w})$ $(u_1, \dots, u_i, \dots, u_k) \longrightarrow (u_1, \dots, u_i^w, \dots, u_k)$, $w \in G$.

Transformations R_{ij}, L_{ij}, I_i are also called *elementary Nielsen transformations.* A composition of finitely many elementary Andrews-Curtis transformations is called an *AC-transformation.* Likewise a composition of elementary Nielsen transformations is a *Nielsen transformation.* Clearly AC transformations are invertible. The group they generate is $FAC_k(G)$, the *full Andrews-Curtis group* of G of rank k. To define the original *Andrews-Curtis group* $AC_k(G)$, denote by $N_k(G)$ the set of all k-tuples in G^k which generate G as a normal subgroup (here we consider only such k that $N_k(G) \neq \emptyset$). Again, every elementary AC-transformation induces a bijection on $N_k(G)$ and the subgroup of $Sym(N_k(G))$ generated by these bijections is the AC group of G, denoted by $AC_k(G)$. The AC conjecture can be stated as follows: for every free group F of rank $k \geq 2$ the Andrews-Curtis group $AC_k(F)$ acts transitively on the set $N_k(F)$. The set $N_k(G)$ can be quite complex, for example, it is known that there is no algorithm to decide whether a group given by a finite presentation is trivial or not [1, 24]. Whether such an algorithm exists for balanced presentations is an open and difficult problem (see [5]). It follows that the set $N_k(F)$ is computably enumerable, but it is not known if it is computable. On the other hand, if G is finitely generated with decidable word problem then the set G^k is computable. In particular, the set F^k is computable. So it might be easier to study the group $FAC_k(G)$ then $AC_k(G)$. Observe, that the restriction of a bijection $\alpha \in FAC_k(G)$ onto the set $N_k(G)$ gives a bijection $\bar{\alpha} \in AC_k(G)$. It is easy to see that the map $\alpha \to \bar{\alpha}$ gives rise to an epimorphism

$$\lambda_{G,k} \colon FAC_k(G) \to AC_k(G).$$

General Problem 1. *For a given group G and $k \geq 2$, describe the kernel of the epimorphism $\lambda_{G,k} \colon FAC_k(G) \to AC_k(G)$.*

Below we address this problem for torsion-free hyperbolic groups G.

The main result of the paper (proved in Section 4) is as follows.

Theorem 2.1. *Let G be a torsion-free non-elementary hyperbolic group. Then $FAC_k(G)$ acts faithfully on every nontrivial orbit in G^k (the trivial orbit consists of the k-tuple $(1, 1, \ldots, 1)$).*

Corollary 2.2. *Let G be a non-elementary torsion-free hyperbolic group. Then for any $k \geq 2$ $\lambda_{G,k} \colon FAC_k(G) \to AC_k(G)$ is an isomorphism.*

Note that in [25] Roman'kov independently proved that $\lambda_{G,k}$ is an isomorphism for free nonabelian groups $G = F_k$ of rank $k \geq 2$.

3. Equations

We require some results about equations over hyperbolic groups. Throughout this section G stands for a non-elementary torsion free hyperbolic group; for example a finitely generated nonabelian free group. Standard references for background on hyperbolic groups are [12, 3, 9, 10].

We state some well-known properties of G as a lemma.

Lemma 3.1. *Let G be a torsion-free non-elementary hyperbolic group.*

(1) *Let H be the centralizer in G of a non-identity elements. Then H is cyclic and malnormal i.e., $H^x \cap H = 1$ if $x \in G - H$.*
(2) *If F is free of finite rank, then the free product $G * F$ is non-elementary torsion-free hyperbolic.*
(3) *G satisfies the big powers property* [21]. *In other words for any sequence $v_1, \ldots, v_n \in G$ such that v_i does not commute with v_{i+1} for $1 \leq i < n$, there exists an integer m such that*

$$v_1^{r_1} \cdots v_n^{r_n} \neq 1 \text{ whenever } r_i \geq m \text{ for all } i.$$

Now we consider equations over G.

Definition 3.2. An equation $E(x_1, \ldots, x_m)$ over G is an element of the free product $G * X$ where X is freely generated by $\{x_1, \ldots, x_m\}$. A tuple $g_1, \ldots g_m$ of elements of G is a solution to E if the unique homomorphism $G * X \to G$ which is the identity on G and sends each x_i to g_i maps E to the identity.

Each equation $E(x_1, \ldots, x_m)$ may be written as $\mathrm{E} = a_0 x_{i_1}^{d_1} a_1 \cdots x_{i_n}^{d_n} a_n$, where the a_j's are elements of G, and the d_j's are nonzero integers. Without loss of generality we may assume that if $a_j = 1$ for some j with $0 < j < n$ then $x_{i_j} \neq x_{i_{j+1}}$. In addition since conjugation of an equation does not change the solution set, we may suppose $a_0 = 1$. Thus we have

$$E = x_{i_1}^{d_1} a_1 x_{i_2}^{d_2} a_2 \cdots x_{i_n}^{d_n} a_n \text{ with } x_{i_j} \neq x_{i_{j+1}} \text{ if } a_j = 1. \tag{3.1}$$

Theorem 3.3. *Let $E(x_1, \ldots, x_m)$ be an equation over a non-elementary torsion-free word-hyperbolic group G. If all m-tuples in G are solutions to E, then $E = 1$ in $G * X$.*

Proof. We argue by induction on n. The cases $n = 0, 1$ are left to the reader, so we may assume $n \geq 2$.

By our hypothesis, substituting 1 for all the x_j's yields $a_1 \cdots a_n = 1$ in G. It follows that we may rewrite Equation (3.1) as a product of conjugates of powers.

$$E = x_{i_1}^{d_1} a_1 x_{i_2}^{d_2} a_1^{-1} a_1 a_2 x_{i_3}^{d_3} (a_1 a_2)^{-1} \cdots (a_1 \cdots a_{n-1}) x_{i_n}^{d_n} (a_1 \cdots a_{n-1})^{-1} \tag{3.2}$$

Define $u_1(x_{i_1}) = x_{i_1}^{d_1}$, $u_2(x_{i_2}) = a_1 x_{i_2}^{d_2} a_1^{-1}$u $u_3(x_{i_3}) = a_1 a_2 x_{i_3}^{d_3} (a_1 a_2)^{-1}$, etc. The group G satisfies the big powers condition (see Lemma 3.1) hence for substitutions of elements $g_{i_j}^{r_j} \in G$ for x_{i_j} either

$$E(g_{i_1}^{r_1}, g_{i_2}^{r_2}, \ldots, g_{i_n}^{r_n}) \neq 1$$

for all sufficiently large integers r_j or for some $j < n$, $u_j(g_{i_j})$ and $u_{j+1}(g_{i_{j+1}})$ commute. In the first case there are many tuples $g_{i_1}^{r_1}, g_{i_2}^{r_2}, \ldots, g_{i_n}^{r_n}$ which are not solutions to our equation E, so we may assume that for some $j < n$, $u_j(g_{i_j}^{r_j})$ and $u_{j+1}(g_{i_{j+1}}^{r_{j+1}})$ commute.

In other words

$$(a_1 \cdots a_{j-1}) g_{i_j}^{r_j d_j} (a_1 \cdots a_{j-1})^{-1}$$

commutes with

$$(a_1 \cdots a_j) g_{i_{j+1}}^{r_{j+1} d_{j+1}} (a_1 \cdots a_j)^{-1}$$

whence $g_{i_j}^{r_j d_j}$ commutes with $a_j g_{i_{j+1}}^{r_{j+1} d_{j+1}} a_j^{-1}$. By Lemma 3.1, the group G is commutative transitive (since all proper centralizers are commutative) hence g_{i_j} and $a_j g_{i_{j+1}} a_j^{-1}$ commute. But clearly we can choose $g_{i_j} \in G$ such that g_{i_j} and $a_j g_{i_{j+1}} a_j^{-1}$ do not commute. Thus there is a substitution $x_i \to g_i^{r_i}$ which does not yield 1 in the equation $E = 1$. □

Remark 3.4. Theorem 3.3 shows that, in terms of algebraic geometry over groups, the radical of the affine space G^n for non-elementary torsion-free hyperbolic groups G is trivial.

Remark 3.5. The proof of Theorem 3.3 shows that the result holds for any group G that satisfy the following conditions:

1) G is CSA, i.e., centralizers of non-trivial elements are abelian and malnormal; there are many such groups which are not hyperbolic (see, for example, [20, 11, 13])
2) G satisfies the big powers condition (see examples in [21, 15]);
3) For any finite subset of non-trivial elements $A \subseteq G$, there is an element $g \in G$ such that $[a, g] \neq 1$ for every $a \in A$.

4. Proof of Theorem 2.1

Notation. When the value of k is irrelevant, we will write $FAC(G)$ in place of $FAC_k(G)$. In addition we will abbreviate $(u_1, \ldots, u_k)$ to $\vec{u}$.

Notice that for each $\alpha \in FAC(G)$, $\alpha(\vec{u})$ is computed by a fixed sequence of elementary AC-moves on $\vec{u}$. Performing the same sequence on a tuple of indeterminants $\vec{x} = (x_1, \ldots, x_k)$ yields group words $(W_1, \ldots, W_k)$ in the free product $G * X$. Here X is the free group over $\{x_1, \ldots, x_k\}$. We record this observation as a lemma.

Lemma 4.1. *For every $\alpha \in FAC(G)$ there are group words $W_i(x_1, \ldots, x_k)$ for $1 \le i \le k$ over indeterminants $x_1, \ldots, x_k$, such that*

$$\alpha(u_1, \ldots, u_k) = (W_1(u_1, \ldots, u_k), \ldots, W_k(u_1, \ldots, u_k)).$$

Let G be a torsion-free non-elementary hyperbolic group, and suppose $\alpha \in FAC(G)$ fixes all elements in the orbit of $\vec{u} = (u_1, \ldots, u_k)$. Without loss of generality we may assume $u_i \neq 1$ for all i. It suffices to show that if $\alpha \in FAC(G)$ fixes all conjugates of $\vec{u}$, i.e., all sequences $(u_1^{h_1}, \ldots, u_k^{h_k})$ as the h_i's run over all elements of G, then $\alpha = 1$.

By Lemma 4.1 there are group words $W_i(x_1, \ldots, x_k)$ over $G * X$ such that

$$\alpha(v_1, \ldots, v_k) = (W_1(v_1, \ldots, v_k), \ldots, W_k(v_1, \ldots, v_k))$$

for all $\vec{v} \in G^k$. By hypothesis $W_i(u_1^{d_1}, \ldots, u_k^{d_k}) = u_i^{d_i}$ for all $d_1, \ldots, d_k \in G$.

Since $G * X$ is non-elementary torsion-free hyperbolic, it follows from Theorem 3.3 that

$$W_i(u_1^{x_1}, \ldots, u_k^{x_k}) = u_i^{x_i} \text{ in } G * X. \tag{4.1}$$

By properties of free products, there is an endomomorphism $f\colon G * X \to G * X$ which is the identity on G and maps each x_i to $u_i^{x_i}$. A straightforward argument shows that f is injective. Hence Equation 4.1 implies $W_i(x_1, \ldots, x_k) = x_i$. Thus $\alpha = 1$ as desired.

5. Open Problems

Here we collect some open problems on AC-groups $AC_k(G)$. Some of them are new, others appear in various presentations, preprints, or papers.

General Problem 2. *Study groups $FAC_k(G)$ and $AC_k(G)$ for different platform groups G.*

More particular problems are listed below.

Problem 5.1. Which groups $AC(G)$ are finitely presentable?

Note, that in [25] Roman'kov showed that the group $AC_2(F_2)$, where F_2 is a free group of rank 2, is not finitely presented. This brings the following question for free groups F_k of rank k.

Problem 5.2. Is it true that the groups $AC_k(F_k)$ are finitely presented for $k \ge 3$?

Problem 5.3. Does the conclusion of Theorem 2.1 hold for partially commutative groups?

Problem 5.4. Find "good" (quasi-geodesic) normal forms of elements in $AC_k(F_k)$

A solution to Problem 5.4 will enhance efficacy of search for counterexamples to the ACC.s

Problem 5.5. For which k does the group $AC_k(F_k)$ have Kazhdan property (T)?

Positive solution to this problem would explain why the analog of the product replacement algorithm for generators of normal subgroups in black-box groups works rather well (see [6]).

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