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AN AXIOMATIZATION FOR THE UNIVERSAL THEORY OF THE HEISENBERG GROUP

ANTHONY M. GAGLIONE AND DENNIS SPELLMAN

Departmant of Mathematics, U.S. Naval Academy, Annapolis, MD 21402
e-mail address: agagaglione@aol.com

Department of Statistics, Temple University, Philadelphia, PA 19122
e-mail address: dennisspellman1@aol.com

This paper is dedicated to the memory of Benjamin Fine.

Abstract. The Heisenberg group, here denoted H, is the group of all 3×3 upper unitriangular matrices with entries in the ring $\mathbb{Z}$ of integers. A.G. Myasnikov posed the question of whether or not the universal theory of H, in the language of H, is axiomatized, when the models are restricted to H-groups, by the quasi-identities true in H together with the assertion that the centralizers of noncentral elements be abelian. Based on earlier published partial results we here give a complete proof of a slightly stronger result.

1. Introduction

A (multiplicatively written) group G is *commutative transitive*, briefly CT, provided the relation of commutativity is transitive on $G\backslash\{1\}$; equivalently, provided the centralizer of every element $g \neq 1$ is abelian.

Noncyclic free groups are universally equivalent, even elementarily equivalent. Myasnikov and Remeslennikov [MR] proved that their universal theory is axiomatized by the quasi-identities they satisfy together with commutative transitivity. Fixing a noncyclic free group F, they proved the analogous result in the language of F when the models are restricted to F-groups.

Let A be a countably infinite set well-ordered as

$$\{a_1, a_2, ..., a_n, ...\} = \{a_{n+1} : \ n < \omega\}$$

where ω is the first limit ordinal which we take as the set of nonnegative integers provided with its natural order. Let $F_\omega(\mathcal{N}_2)$ be the group free in the variety of all 2-nilpotent groups on the generators A. For each integer $n \geq 2$ let $F_n(\mathcal{N}_2)$ be the subgroup of $F_\omega(\mathcal{N}_2)$ generated (necessarily freely) by the initial segment $\{a_1, a_2, ..., a_n\}$ of A. The *Heisenberg group* is the

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group H of all 3×3 upper unitriangular matrices with entries in the ring $\mathbb{Z}$ of integers. It is free in $\mathcal{N}_2$ on the generators

$$a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(See [4]). We take the liberty of identifying $F_2(\mathcal{N}_2)$ with H.

Now $F_\omega(\mathcal{N}_2)$ is *discriminated* by the family of retractions $F_\omega(\mathcal{N}_2) \to H$. That means that given finitely many elements $f_1, ..., f_k \in F_\omega(\mathcal{N}_2)\backslash\{1\}$ there is a retraction $F_\omega(\mathcal{N}_2) \to H$ which doesn't annihilate any of them. (H *discriminates* $\mathcal{N}_2$ in the sense of Hanna Neumann [8].) From this it follows that $F_n(\mathcal{N}_2)$, $n \geq 2$, are universally equivalent; moreover, since the discrimination is done by retractions, they are universally equivalent in the language of H. (See [3]).

Let CT(1) or *noncentral commutative transitivity*, briefly NZCT, be the property that the relation of commutativity be transitive on $G\backslash Z(G)$ where $Z(G)$ is the center of G. Equivalently, NZCT asserts that the centralizers of noncentral elements are abelian. A special case of a question posed by A.G. Myasnikov is whether or not the universal theory of noncyclic free 2-nilpotent groups is axiomatized by the quasi-identities they satisfy together with NZCT and whether or not that theory in the language of H is so axiomatized when the models are restricted to H-groups.

In this paper, in the case of the language of H, we answer that question in the positive. In fact we prove a slightly stronger result. The remainder of this paper contains four additional sections. In Section 2 we fix definitions and notation. In Section 3 we prove the main result. In Section 4 we ponder but do not settle the question in the language without parameters from H. Finally in Section 5 we suggest problems for future research.

Before closing the introduction we note that the variety $\mathcal{A}^2$ of metabelian groups is discriminated by its rank 2 free group. From this it follows that the noncyclic free metabelian groups are universally equivalent. It is worth mentioning in passing that Remeslenikov and Stohr proved in [9] that the universal theory of the noncyclic free metabelian groups is axiomatized by the quasi-identities they satisfy together with commutative transitivity.

2. Pedantic Preliminaries

Let L_0 be the first order language with equality containing a binary operation symbol $\cdot$, a unary operational symbol $^{-1}$ and a constant symbol $\widehat{1}$. If G is a (multiplicatively written) group $L_0[G]$ is obtained from L_0 by adjoining names $\widehat{g}$ for the elements $g \in G\backslash\{1\}$ as new constant symbols. We find it convenient to commit the "abuses" of identifying $\widehat{g}$ with g for all $g \in G$ and replacing $\cdot$ with juxtaposition. Moreover, we find it convenient to write an inequation $\sim (s = t)$ as $s \neq t$.

An *identity*, in $L_0[G]$ (Note: $L_0 = L[\{1\}]$) is a universal sentence of the form $\forall \overline{x}\ (s(\overline{x}) = t(\overline{x}))$ where $\overline{x}$ is a tuple of variables and $s(\overline{x})$ and $t(\overline{x})$ are terms of $L_0[G]$. Examples of identities are the group axioms, namely:

$$\begin{aligned} \forall x_1, x_2, x_3\ ((x_1x_2)x_3 &= x_1(x_2x_3)) \\ \forall x\ (x1) &= x \\ \forall x\ (x\ x^{-1}) &= 1). \end{aligned}$$

A *quasi-identity* of $L_0[G]$ is universal sentence of the form

$$\forall\ \overline{x}\ \left(\bigwedge_{i=1}^{k} (\ s_i(\ \overline{x}) = t_i(\ \overline{x})) \rightarrow\ (s(\ \overline{x}) = t(\ \overline{x})) \right)$$

where $\overline{x}$ is a tuple of variables and the $s_i(\ \overline{x})$, $t_i(\ \overline{x})$, $s(\ \overline{x})$ and $t(\ \overline{x})$ are terms of $L_0[G]$. Every identity is equivalent to a quasi-identity since $\forall \overline{x}\ (s(\overline{x}) = t(\overline{x}))$ is equivalent to

$$\forall \overline{x}, y\ ((y = y) \rightarrow\ (s(\overline{x}) = t(\overline{x}))).$$

In particular, the group axioms are equivalent to quasi-identities.

If G is a group and we let $\mathcal{Q}^0(G)$ be the set of all quasi-identities of L_0 true in G and $\mathcal{Q}(G)$ be the set of all quasi-identities of $L_0[G]$ true in G. We view the group axioms as contained in $\mathcal{Q}^0(G) \subseteq \mathcal{Q}(G)$. We let $Th^0_\forall(G)$ be the set of all universal sentences of L_0 true in G and $Th_\forall(G)$ be the set of all universal sentences of $L_0[G]$ true in G. Note that quantifier free sentences are viewed as special cases of universal sentences. In particular, the *diagram* of G, briefly $\operatorname{diag}(G)$, consisting of the atomic and negated atomic sentences of $L_0[G]$ true in G is a set of universal sentences of $L_0[G]$.

A *G-group* Γ is a model of the group axioms and $\operatorname{diag}(G)$. That is equivalent to the group Γ containing a distinguished copy of G as a subgroup. A *G-polynomial* is a group word on the elements of G and variables. (If you like, an element of the free product $G * \langle x_1, ..., x_n;\ \rangle$ for some n.) Note that, modulo the group axioms, every identity of L_0 ($L_0[G]$) is equivalent to one of the form $\forall \overline{x}\ \ (w(\overline{x}) = 1)$ where $w(\overline{x})$ is a group word (G-polynomial) and every quasi-identity of L_0 ($L_0[G]$) is equivalent to one of the form

$$\forall\ \overline{x}\ \left(\bigwedge_{i=1}^{k} (\ u_i(\ \overline{x}) = 1) \rightarrow\ (w(\ \overline{x}) = 1) \right)$$

where the $u_i(\overline{x})$ and $w(\ \overline{x})$ are group words (G-polynomials).

In this paper by "ring" we shall always mean commutative ring with multiplicative identity $1 \neq 0$. Subrings are required to contain 1 and homomorphisms are required to preserve 1. A ring R is *residually*-$\mathbb{Z}$ provided, given $r \in R\backslash\{0\}$, there is a homomorphism $R \to \mathbb{Z}$ which does not annihilate r. This forces R to have characteristic zero and we identify the minimum subring of R with $\mathbb{Z}$. Hence, we view R as separated by retractions $R \to \mathbb{Z}$. A ring is *locally residually*-$\mathbb{Z}$ provided every finitely generated subring is residually-$\mathbb{Z}$. Being locally residually-$\mathbb{Z}$ is equivalent to being a model of the quasi-identities of ring theory true in $\mathbb{Z}$ (See [4]).

It was proven in [4] that every model of $\mathcal{Q}(H) \cup \operatorname{diag}(H)$ H-embeds in $UT_3(R)$ for some locally residually-$\mathbb{Z}$ ring R. (Conversely every H-subgroup of such a $UT_3(R)$ is a model of $\mathcal{Q}(H) \cup \operatorname{diag}(H)$.) Here an *$H$-embedding* is an embedding which is the identity on H. (The meanings of *H-subgroup* and *H-homomorphism* in the category of H-groups are readily apparent.)

Let G be a group and $g \in G$. We let $C_G(g)$ be the centralizer of g in G. NZCT is the following universal sentence of L_0 :

$$\forall x_1, x_2, x_3, y\ \ ((([x_2, y] \neq 1) \wedge ([x_1, x_2] = 1) \wedge ([x_2, x_3] = 1)) \rightarrow ([x_1, x_3] = 1)).$$

A group G satisfies NZCT if and only if $C_G(g)$ is abelian for all $g \in G\backslash Z(G)$. The following quasi-identity of $L_0[H]$ holds in H.

$$\forall x, z\ \ ((([z, a_1] = 1) \wedge ([a_2, z] = 1)) \rightarrow ([z, x] = 1)).$$

It follows that, if G is any model of $\mathcal{Q}(H) \cup \mathrm{diag}(H)$, then $C_G(a_1) \cap C_G(a_2) = Z(G)$. In particular, if G additionally satisfies NZCT, then the following universal sentence τ of $L_0[H]$ is a consequence

$$\forall x_1, x_2 \left(\begin{array}{c} (([x_2, x_1] = 1) \wedge ([a_2, x_2] = 1) \wedge ([x_1, a_1] = 1) \ \rightarrow \\ (([x_2, a_1] = 1) \vee ([a_2, x_1] = 1)) \end{array} \right).$$

We shall prove in the next section that $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ axiomatizes $Th_{\forall}(H)$.

A *primitive sentence* of $L_0[H]$ is (modulo the group axioms) an existential sentence of the form $\exists\ \overline{x} \left(\bigwedge_i (p_i(\ \overline{x}) = 1) \ \wedge (\bigwedge_j (q_j(\ \overline{x}) \neq 1) \right)$ where the $p_i(\ \overline{x})$ and $q_j(\ \overline{x})$ are H-polynomials.

By writing the matrix of an existential sentence in disjunctive normal form one sees that every existential sentence of $L_0[H]$ is equivalent (modulo the group axioms) to a disjunction of primitive sentences and so holds in an H-group if and only if at least one disjunct does.

Assume momentarily that there is a universal sentence φ of $L_0[H]$ which holds in H but is not a consequence of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$. Then its negation $\sim \varphi$ must hold in some model G of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$. Since $\sim \varphi$ is equivalent to an existential sentence of $L_0[H]$ there must be a primitive sentence

$$\exists x_1, ..., x_k \left(\bigwedge_i (p_i(\ a_1, a_2, x_1, ..., x_k) = 1) \ \wedge (\bigwedge_j (q_j(a_1, a_2, x_1, ..., x_k) \neq 1) \right)$$

of $L_0[H]$ which holds in G but is false in H. Let the assignment $x_\lambda \mapsto g_\lambda$, $1 \leq \lambda \leq k$, verify the above sentence in G. Then the finitely generated H-subgroup $G_0 = \langle a_1, a_2, g_1, ...g_k \rangle$ of G also satisfies the above primitive sentence and, since universal sentences of $L_0[H]$ are preserved in H-subgroups, G_0 is a model of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$. Hence, if a counterexample exists, then so would a finitely generated counterexample exist. So it suffices to prove the result for finitely generated models. We shall find it convenient to prove, more generally, that the result holds for models G such that the quotient $\overline{G} = G/Z(G)$ is finitely generated.

3. The Lame Property and the Universal Theory of H

It was shown in [5] using a characterization due to Mal'cev [7] that a model E of $\mathcal{Q}(H) \cup \mathrm{diag}(H)$ is of the form $UT_3(R)$ for some locally residually-$\mathbb{Z}$ ring R if and only if E satisfies the following universal-existential sentence

σ of $L_0[H]$:

$$\forall x_1, x_2 \exists y_1, y_2 \left(\begin{array}{c} ([y_1, a_1] = 1) \wedge ([a_2, y_2] = 1) \wedge ([x_2, x_1] = [y_2, a_1]) \\ \wedge ([x_2, x_1] = [a_2, y_1]) \end{array} \right).$$

In other words, for each commutator $[g_2, g_1]$, each of the systems

$$S \left\{ \begin{array}{c} [a_2, y] = 1 \\ [y, a_1] = [g_2, g_1] \end{array} \right.$$

and

$$T \left\{ \begin{array}{c} [x, a_1] = 1 \\ [a_2, x] = [g_2, g_1] \end{array} \right.$$

has a solution in E.

Mimicking the construction of an existentially closed extension (See e.g. Hodges [H]) we may embed a model G of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ into $UT_3(R)$, If we could preserve τ at each step then, since universal sentences are preserved in direct unions, $UT_3(R)$ would also satisfy τ. If $UT_3(R)$ satisfies τ, then R is an integral domain. To see this let $(r, s) \in R^2$ and suppose $rs = 0$. Let $y = \begin{bmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$. Then $[y, x] = 1$ and $[a_2, y] = 1$ and $[x, a_1] = 1$. Moreover, $[y, a_1] = \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[a_2, x] = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since $UT_3(R)$ satisfies τ, $r = 0$ or $s = 0$ and R indeed is an integral domain.

A residually-$\mathbb{Z}$ ring R is *ω-residually-$\mathbb{Z}$* provided it is discriminated by the family of retractions $R \to \mathbb{Z}$. That is, given finitely many nonzero elements of R, there is a retraction $R \to \mathbb{Z}$ which does not annihilate any of them. That is equivalent to R being an integral domain. Suppose first that R is an integral domain. Let $r_1, ..., r_n$ be finitely many nonzero elements of R. Then the product $r = r_1 \cdots r_n \neq 0$ and there is a retraction $\rho : R \to \mathbb{Z}$ which does not annihilate r so cannot annihilate and of $r_1, ..., r_n$. Conversely, if R is ω-residually-$\mathbb{Z}$ and r and s are nonzero elements of R, then there is a retraction $\rho : R \to \mathbb{Z}$ such that $\rho(r) \neq 0$ and $\rho(s) \neq 0$; so, $\rho(rs) = \rho(r)\rho(s) \neq 0$ and hence $rs \neq 0$ and R must be an integral domain. From this it follows that a locally residually-$\mathbb{Z}$ ring R is locally ω-residaully-$\mathbb{Z}$ if and only if R is an integral domain. So, if $UT_3(R)$ satisfies τ, then R is locally ω-residually-$\mathbb{Z}$.

Given $G \leq_H UT_3(R)$, G is the direct union $\lim\limits_{\rightarrow} (G \cap UT_3(R_0))$ as R_0 varies over the finitely generated subrings of R. Ring retractions $R_0 \to \mathbb{Z}$ induce group retractions $G \cap UT_3(R_0) \to H$. From this it follows that each $G \cap UT_3(R_0)$ is discriminated by the family of group retractions $G \cap UT_3(R_0) \to H$ and hence each $G \cap UT_3(R_0)$ is a model of $Th_\forall(H)$. Since universal sentences are preserved in direct unions and $G = \lim\limits_{\rightarrow} (G \cap UT_3(R_0))$ we have that G is a model of $Th_\forall(H)$. The bottom line is that we would be finished if we could construct an overgroup $UT_3(R)$ in such a way that τ is preserved at each step of the construction.

Let us forget momentarily about this particular R and consider a possible property that a representation $G \leq_H UT_3(R)$ might satisfy.

Definition 3.1. Let G be a model of $\mathcal{Q}(H) \cup \mathrm{diag}(H)$ and let $G \leq_H UT_3(R)$ where R is locally residually-$\mathbb{Z}$. We say the representation satisfies the *Lame Property* provided, for each $g = \begin{bmatrix} 1 & g_{12} & g_{13} \\ 0 & 1 & g_{23} \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_1) \cup C_G(a_2)$, either $g_{12}^2 + g_{23}^2 = 0$ or $g_{12}^2 + g_{23}^2$ is not a zero divisor in R.

Lemma 3.2. *Given a model G of $\mathcal{Q}(H) \cup$ diag(H) and a representation $G \leq_H UT_3(R)$ where R locally residually-$\mathbb{Z}$. The representation satisfies the Lame Property if and only if it satisfies the conjunction of the following two conditions:*

(1.) *For all* $y = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_2) \backslash Z(G)$, y_{12} *is not a zero divisor in* R.

(2.) For all $x = \begin{bmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_1)\backslash Z(G)$, *$x_{23}$ is not a zero divisor in R.*

Proof. Suppose the representation satisfies the Lame Property. Let $y = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_2)\backslash Z(G)$. Then $y_{12} \neq 0$.

The quasi-identity $\forall x\ ((x^2 = 0) \to (x = 0))$ holds in $\mathbb{Z}$; hence, it is true in R and $y_{12}^2 \neq 0$. If $r \neq 0$ annihilates y_{12} then $r(y_{12}^2 + y_{23}^2) = ry_{12}^2 = 0$ contradicting $y_{12}^2 = y_{12}^2 + y_{23}^2$ is not a zero divisor in R. The contradiction shows that the representation satisfies (1.). Similarly, the representation satisfies (2.).

Now suppose that representation satisfies (1.) and (2.). Let $g = \begin{bmatrix} 1 & g_{12} & g_{13} \\ 0 & 1 & g_{23} \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_1) \cup C_G(a_2)$. $\mathbb{Z}$ satisfies each of the quasi-identities

$$\begin{aligned} \forall x, y\ ((x^2 + y^2 &= 0) \to (x = 0)) \text{ and} \\ \forall x, y\ (x^2 + y^2 &= 0) \to (y = 0)); \text{ so,} \end{aligned}$$

they must hold in R. Hence, if $g_{12}^2 + g_{23}^2 \neq 0$, then either $g_{12} \neq 0$ or $g_{23} \neq 0$.

Suppose there were an $r \neq 0$ which annihilates $g_{12}^2 + g_{23}^2$. From $r(g_{12}^2 + g_{23}^2) = 0$ we get $(rg_{12})^2 + (rg_{23})^2 = r^2(g_{12}^2 + g_{23}^2) = 0$. Hence $rg_{12} = 0$ and $rg_{23} = 0$. If $g_{12} \neq 0$ (1.) is contradicted while if $g_{23} \neq 0$ (2.) is contradicted. The contradiction shows that the conjunction of (1.) and (2,) implies the Lame Property.

We next note that if the representation $G \leq_H UT_3(R)$ satisfies the Lame Property, then G satisfies τ.

For suppose $[a_2, y] = 1$ so $y = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_2)$, $[x, a_1] = 1$ so $x = \begin{bmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} \in C_G(a_1)$ and $[y, x] = \begin{bmatrix} 1 & 0 & y_{12}x_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $y_{12}x_{23} = 0$. But if $y_{12} \neq 0$, then $x_{23} = 0$ otherwise (1.) is contradicted while if $x_{23} \neq 0$, then $y_{12} = 0$ otherwise (2.) is contradicted. It follows that either $[y, a_1] = 1$ or $[a_2, x] = 1$ so τ holds in G. For a fixed representation $G \leq_H UT_3(R)$ satisfying the Lame Property is a sufficient condition for τ to hold in G; however, it is not a necessary condition for G to satisfy τ. (None the less we shall subsequently see that having at least one representation satisfying the Lame Property is necessary and sufficient.)

The result (proven in [5]) that every 3-generator model of $\mathcal{Q}(H) \cup \text{diag}(H)$ is already a model of the $Th_\forall(G)$ provides a treasure trove of counterexamples.

Let G be a model of $\mathcal{Q}(H) \cup \text{diag}(H)$ and let R be a locally residually-$\mathbb{Z}$ ring. We say that R is *appropriate for G* provided

(1) $G \leq_H UT_3(R)$ and

(2) R is generated by the entries of the elements of G.

Now $R = \mathbb{Z}\times\mathbb{Z} = \mathbb{Z}e_1 + \mathbb{Z}e_2$ where $e_1 = (1,0)$ and $e_2 = (0,1)$ is residually-$\mathbb{Z}$. Consider the 3-generator subgroup $G \leq_H UT_3(R)$ generated by

$$a_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix},\ a_2 = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ and } b = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & e_1\\ 0 & 0 & 1\end{bmatrix}.$$

Since $1 - e_1 = e_2$, $R = \mathbb{Z}\times\mathbb{Z}$ is generated by the entries of the elements of G. Since $e_1e_2 = 0$, e_1 is a zero divisor in R.

Now $b \in C_G(a_1)\backslash Z(G)$ and $b_{23} = e_1$ is a zero divisor in R. Hence, the representation violates the Lame Property. Every 3-generator model of $\mathcal{Q}(H)\cup\mathrm{diag}(H)$ is already a model of $Th_\forall(H)$. So this $G = \langle a_1, a_2, b\rangle$ satisfies τ. Now this G is obtained from H by extending $C_H(a_1)$ introducing a new parameter. (It is a rank 1 centralizer extension relative to the category $\boldsymbol{N}_2$ of 2-nilpotent groups.)

Let θ be an indeterminate over $\mathbb{Z}$. Then the polynomial ring $\mathbb{Z}[\theta]$ is residually-$\mathbb{Z}$ and we could have just as well embedded this G into $UT_3(\ \mathbb{Z}[\theta]\ )$ as the subgroup generated by

$$a_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix},\ a_2 = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ and } b = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \theta\\ 0 & 0 & 1\end{bmatrix}.$$

Since θ is an entry of an element of G, $\mathbb{Z}[\theta]$ is also appropriate for G. Since $\mathbb{Z}[\theta]$ is an integral domain, the representation does satisfy the Lame Property.

Anticipating an application to be used later in this paper, suppose G_0 is a model of $\mathcal{Q}(H)\cup\mathrm{diag}(H)$ and let a_i be a free generator of H, $i \in \{1,2\}$. Suppose G is obtained from G_0 by extending $C_{G_0}(a_i)$. That is

$$G = \langle G_0, t;\ rel(G_0),\ [t, C_{G_0}(a_i)] = 1\rangle_{\mathcal{N}_2}.$$

Using a big powers argument, we get a discriminating family of retractions $G \to G_0$ via

$$\begin{cases} g \mapsto g\ ,\ g \in G_0\\ t \mapsto a_i^n\ ,\ n \in \mathbb{Z}\end{cases}.$$

It follows that G is universally equivalent to G_0.

Now suppose we fix a model G_0 of $\mathcal{Q}(H)\cup\mathrm{diag}(H)\cup\{\tau\}$ which admits a representation $G_0 \leq_H UT_3(R)$ satisfying the Lame Property. Let $(g_1, g_2) \in G_0^2$ and suppose the system

$$S\begin{cases} [a_2, y] = 1\\ [y, a_1] = [g_2, g_1]\end{cases}$$

has no solution in G_0. Let $[g_2, g_1] = z = \begin{bmatrix}1 & 0 & z_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$. Let Y be the element $\begin{bmatrix}1 & z_{13} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ of $UT_3(R)$. Then $[a_2, Y] = 1$ and $[Y, a_1] = [g_2, g_1]$ so $Y \notin G_0$.

Let G_1 be the subgroup $\langle G_0, Y\rangle$ of $UT_3(R)$. Collecting and simplifying we see a typical element of G_1 has the form $uY^n\ [Y, w]$ where $n \in \mathbb{Z}$ and $(u, w) \in G_0^2$. The matrix representing this element has the form $\begin{bmatrix}1 & u_{12} + nz_{13} & *\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$. Now suppose $C = \begin{bmatrix}1 & c_{12} + nz_{13} & *\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \in C_{G_1}(a_2)$ and $B = \begin{bmatrix}1 & 0 & *\\ 0 & 1 & b_{23}\\ 0 & 0 & 1\end{bmatrix} \in C_{G_1}(a_1)$. Assume further that

$[C, B] = 1$. Then $\begin{bmatrix} 1 & 0 & (c_{12} + nz_{13})b_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $(c_{12} + nz_{13})b_{23} = 0$. Now, $c_{12} + nz_{13} \neq 0$ and $b_{23} \neq 0$. Moreover, B has the form $u[Y, w]$ with $u \in C_{G_0}(a_1)$ looking like $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$. Since b_{23} is a zero divisor in R that contradicts that the representation G_0 satisfies the Lame Property. Hence, either $c_{12} + nz_{13} = 0$ or $b_{23} = 0$ so either$[C, a_1] = 1$ or $[a_2, B] = 1$ and G_1 satisfies τ.

Similarly, if the system

$$T \begin{cases} [x, a_1] = 1 \\ [a_2, x] = [g_2, g_1] \end{cases}$$

has no solution in G_0 we can extend to a model of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ in which T has a solution.

Getting back to G_1, suppose the system

$$T \begin{cases} [x, a_1] = 1 \\ [a_2, x] = [g_2, g_1] \end{cases}$$

has no solution in G_1. Suppose further that G_1 admits a representation $G_1 \leq_H UT_3(R)$ satisfying the Lame Property. Then we can extend G_1 to a model G of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ in which T has a solution. We have the chain $G_0 \leq G_1 \leq G$. G is a model of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ in which the system

$$S \cup T \begin{cases} [a_2, y] = 1 \\ [y, a_1] = [g_2, g_1] \\ [x, a_1] = 1 \\ [a_2, x] = [g_2, g_1] \end{cases}$$

has a solution. So, if every model G of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ has at least one representation satisfying the Lame Property, then G embeds in $UT_3(R)$ where R is an integral domain and we are finished. To prove that every model of $\mathcal{Q}(H) \cup \mathrm{diag}(H) \cup \{\tau\}$ admits a representation satisfying the Lame Property, we need some results from model theory.

Let $\mathbb{M}$ be the model class operator. We first paraphrase a result from Bell and Slomson [1].

Proposition 3.3. *Let L be a first order language with equality. Let K be the class of all Lstructures and let $X \subseteq K$. Then there is a set S of sentences of L such that $X = \mathbb{M}(S)$ if and only if X is closed under isomorphism and ultraproducts and $K \backslash X$ is closed under ultraprowers.*

Remark 3.4. The proof in [1] needed the Generalized Continuum Hypothesis. In view of Shelah's [10] improvement of Keisler's ultrapower theorem that hypothesis may be omitted.

The next result may be found in Hodges [H].

Proposition 3.5. *Let L be a first order language with equality. Let T be a set of sentences of L. Let $T_\forall$ be the set of all universal sentences of L which are logical consequences of T. Then $\mathbb{M}(T_\forall)$ consists of all L-substructures of models of T.*

Using Proposition 1, a striaghtforward but tedious verification reveals that the class of all models of $\mathcal{Q}(H) \cup \text{diag}(H) \cup \{\tau\}$ which admit a representation satisfying the Lame Property is first order. Moreover, since this class is closed under H-subgroups, it has (as an application of Proposition 2) a set Φ of universal axioms in $L_0[H]$. Now $\mathbb{M}(\mathcal{Q}(H) \cup \text{diag}(H) \cup \{\tau\}) \supseteq \mathbb{M}(\Phi)$. We would be finished if we could prove equality. It will suffice to establish the result for finitely generated models. We shall prove the result more generally for models G of $\mathcal{Q}(H) \cup \text{diag}(H) \cup \{\tau\}$ such that the quotient $\overline{G} = G/Z(G)$ is finitely generated.

Theorem 3.6. *Every model of $\mathcal{Q}(H) \cup diag(H) \cup \{\tau\}$ admits a representation satisfying the Lame Property.*

Corollary 3.7. *$\mathcal{Q}(H) \cup diag(H) \cup \{\tau\}$ is an axiomatization for $Th_\forall(H)$.*

Corollary 3.8. *$\mathcal{Q}(H) \cup diag(H) \cup \{NZCT\}$ is an axiomatization for $Th_\forall(H)$.*

Proof of Theorem 3.6. Let G be a model of $\mathcal{Q}(H) \cup \text{diag}(H) \cup \{\tau\}$. We may assume without loss of generality that the quotient $\overline{G} = G/Z(G)$ is finitely generated. Let $\overline{C_i} = C_G(a_i)\ /Z(G),\ i = 1, 2$. Since $\langle \overline{a_i}\ \rangle = \langle a_i Z(G) \rangle \subseteq \overline{C_i}$, $\overline{C_i}$ has finite rank at least 1, $i = 1, 2$.

Thus, $\text{rank}(\overline{C_1}) + \text{rank}(\overline{C_2}) \geq 2$. Define the *$C$-rank* of G to be

$$\text{rank}(\overline{C_1}) + \text{rank}(\overline{C_2}) - 1.$$

The proof will proceed by induction on the C-rank.

Suppose first that the C-rank of G is 1. That forces

$$\text{rank}(\overline{C_1}) = 1 = \text{rank}(\overline{C_2})$$

and $\overline{C_i} = \langle \overline{a_i}\ \rangle = \langle a_i Z(G) \rangle$, $i = 1, 2$. Let $G \leq_H UT_3(R)$ be any representation where R is locally residually-$\mathbb{Z}$.

It follows from the above that every element of $C_G(a_2) \backslash Z(G)$ looks like $\begin{bmatrix} 1 & m & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $m \in \mathbb{Z} \backslash \{0\}$ and every element of $C_G(a_1) \backslash Z(G)$ looks like $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$ where $n \in \mathbb{Z} \backslash \{0\}$.

Now, for each $k \in \mathbb{Z} \backslash \{0\}$, the quasi-identity $\forall x\ ((kx = 0) \to (x = 0))$ holds in $\mathbb{Z}$. Hence, these quasi-identities hold in R and consequently every representation of G satisfies the Lame Property. The initial step of the induction has been established.

Now suppose G has C-rank $n > 1$ and the result has been established for models with C-rank k with $1 \leq k < n$.

Now let $G \leq_H UT_3(R)$ be any representation of G where R is locally residually-$\mathbb{Z}$. Let us extend G to $\widehat{G}$ by adjoining the elements $\begin{bmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as r varies over R. Since τ depends on the $(1, 2)$ and $(2, 3)$ entries only and since $\widehat{G}$ has the same C-rank as G, we may replace G with $\widehat{G}$. This causes no harm since universal sentences are preserved in subgroups; so G will be a model of Φ whenever $\widehat{G}$ is.

Since the C-rank of $\widehat{G}$ is greater than 1 at least one of

$$\overline{C_i} = C_{\widehat{G}}(a_i)/Z(\widehat{G})\ ,\ i = 1, 2$$

must have rank at least 2. We may assume $\text{rank}(\overline{C_2}) = m \geq 2$. Choose elements $a_2, b_1, ..., b_{m-1} \in C_{\widehat{G}}(a_2)$ which project modulo $Z(\widehat{G})$ to a basis for $\overline{C_2}$. Now let N be the subgroup $\langle b_{m-1}\rangle \cdot Z(\widehat{G})$ of $\widehat{G}$. Since $[\widehat{G}, \widehat{G}] \leq Z(\widehat{G}) \leq N$, N is normal in $\widehat{G}$ and $\widehat{G}/N$ is abelian. Let T be a transversal for N in $\widehat{G}$ so that every element g is uniquely expresses in the form $t(g)\upsilon(g)$ where $t(g) \in T$ and $\upsilon(g) \in N$. Assume further that, for all $(p, q, z) \in \mathbb{Z}^2 \times Z(\widehat{G})$, $t(a_1^p a_2^q z) = a_1^p a_2^q$ and that $t(x) = 1$ for all $x \in N$.

Let G_0 be the subgroup of $\widehat{G}$ generated by the coset representatives of N in $\widehat{G}$ together with $Z(\widehat{G})$. Since b_{m-1} is killed off the C-rank of G_0 is less than that of $\widehat{G}$ and so, by inductive hypothesis, G_0 is a model of Φ.

Examining the general form of a matrix in $\widehat{G}$ and setting that guy equal to the identity matrix, we see that, modulo the law $[x_1, x_2, x_3] = 1$, the only relations in $\widehat{G}$ are consequences of the relations in G_0 and the relations $[C_{G_0}(a_2), b_{m-1}] = 1$. That is, $\widehat{G}$ is the free rank 1 extension of $C_{G_0}(a_2)$ relative to $\boldsymbol{N}_2$ and hence G_0 and $\widehat{G}$ are universally equivalent. Then $\widehat{G}$ is a model of Φ since G_0 is. Since $G \leq \widehat{G}$ and universal sentences are preserved in subgroups, G is a model of Φ. That completes the induction and proves the theorem.

4. The Theory in the Base Language

We wish to ponder whether or not $Q^0(H) \cup \{NZCT\}$ axiomatizes $Th^0_\forall(H)$. To that end let G_0 be a model of $Q^0(H) \cup \{NZCT\}$. We may assume G_0 is finitely generated. Suppose that G_0 is abelian. Then, since models of $Q^0(H)$ are torsion free, G_0 is free abelian of finite rank r. Choose a positive integer n such that $\left(\begin{array}{c} n \\ 2 \end{array}\right) \geq \max\{r, 2\}$. If $G = F_n(\mathcal{N}_2)$ then $[G, G]$ is free abelian of rank $\left(\begin{array}{c} n \\ 2 \end{array}\right)$. It follows that G_0 embeds in G . So every universal sentence of L_0 true in G must also be true in G_0. But G is universally equivalent to H. Therefore G_0 is a model of $Th^0_\forall(H)$. So it now suffices to assume G_0 is a finitely generated nonabelian model of $Q^0(H) \cup \{NZCT\}$. A consequence of a result of Grätzer and Lasker [6] is that the quasivariety generated by H consists of all groups isomorphic to a subgroup of a direct product of a family of ultrapowers of H. View H as $UT_3(\mathbb{Z})$ and taking corresponding direct product of ultrapowers of $\mathbb{Z}$ we get a ring R such that G_0 embeds in $UT_3(R)$. Since quasi-identities are preserved in direct products and ultrapowers, R is a model of the quasi-identities true in $\mathbb{Z}$. That is R is locally residually-$\mathbb{Z}$. Further we may take R to be generated by the entries of a fixed finite set of generators for G_0. Thus, R may be taken finitely generated. Therefore R is residually-$\mathbb{Z}$ and so separated by the family of retractions $R \to \mathbb{Z}$. Let $\overline{G}$ be the subgroup $\langle G_0, H\rangle$ of $UT_3(R)$. The retractions $R \to \mathbb{Z}$ induce group retractions $\overline{G} \to H$ and these separate $\overline{G}$. It follows that $\overline{G}$ is a model of $Q(H) \cup \text{diag}(H)$. Let us keep this $\overline{G}$ in mind as we move on.

Let us say a group is (G_0, H)-*group* if it contains a distinguished copy of each of G_0 and H. The meanings of (G_0, H)-*subgroup* and (G_0, H)-*homomorphism* are readily apparent. A (G_0, H)-*ideal* is the kernel of a (G_0, H)-homomorphism. Equivalently, a (G_0, H)-ideal in a (G_0, H)-group G is a subgroup K normal in G such that $K \cap G_0 = \{1\} = K \cap H$.

Suppose I is a nonempty index set and $(G_i)_{i\in I}$ is a family of (G_0, H)-groups indexed by I. Let G be the direct product $\prod_{i\in I} G_i$. We have the diagonal embeddings

$$\begin{aligned} \alpha &: G_0 \to G \\ \alpha(g)(i) &= g \text{ for all } g \in G_0 \ , \ i \in I \end{aligned}$$

and

$$\begin{aligned} \beta &: H \to G \\ \beta(h)(i) &= h \text{ for all } h \in H \ , \ i \in I. \end{aligned}$$

We view G as a (G_0, H)-group using these diagonal embeddings. Let $\coprod$ be the $\mathcal{N}_2$ free product (See [8]). Let G_1, G_2, and G be 2-nilpotent groups. Let $\varphi_i : G_i \to G$ be a homomorphism $i-1, 2$. Then there is a unique homomorphism $\varphi : G_1 \coprod G_2 \to G$ such that

$$\varphi\mid_{G_i} = \varphi_i \ , \ i = 1, 2.$$

Getting back to $\overline{G}$ and letting $\Gamma = G_0 \coprod H$ we see there is a unique (G_0, H)-homomorphism $\varphi : \Gamma \to \overline{G}$. Note φ is surjective since G_0 and H generate $\overline{G}$. If $K = Ker(\varphi)$, then K is a (G_0, H)-ideal in Γ such that Γ/K is a model of $Q(H)$. Hence the set $\mathbb{K}$ of all (G_0, H)-ideals, K, in Γ such that Γ/K is a model of $Q(H)$ is nonempty. Let $K_0 = \bigcap_{K\in\mathbb{K}} K$. Let $U_H(G_0) = \Gamma/K_0$.

We call $U_H(G_0)$ the *universal H-extension of* G_0. We claim that $U_H(G_0)$ is a (G_0, H)-group which is a model of $Q(H)$ and that if G is any (G_0, H)-group which is a model of $Q(H)$, then there is a unique (G_0, H)-homomorphism $U_H(G_0) \to G$. Put another way, we claim that $U_H(G_0)$ is an initial object in the category whose objects are (G_0, H)-groups which are models of $Q(H)$ and whose morphisms are (G_0, H)-homomorphisms.

From $K \cap G_0 = \{1\} = K \cap H$ for all $K \in \mathbb{K}$ we get $K_0 \cap G_0 = 1 = K_0 \cap H$ as $K_0 \leq K$ for all $K \in \mathbb{K}$. Thus, K_0 is a (G_0, H)-ideal and $U_H(G_0)$ is a (G_0, H)-group.

We get a (G_0, H)-homomorphism

$$\varphi : \Gamma \to \prod_{K\in\mathbb{K}} (\Gamma/K) \text{ via}$$

$\varphi(\gamma) = (\gamma K)_{K\in\mathbb{K}}$ for all $\gamma \in \Gamma$. $K_0 = Ker(\varphi)$. It follows that $U_H(G_0)$ isomorphic to the image of φ. Since quasi-identities of $L_0[H]$ are preserved in direct products and H-subgroups, $U_H(G_0)$ is a model of $Q(H)$.

Now suppose G is any (G_0, H)-group which is a model of $Q(H)$. Since $\Gamma = G_0 \coprod H$ we get a unique homomorphism $\pi : \Gamma \to G$ which restricts to the identity on each of G_0 and H. Then $Ker(\pi) \in \mathbb{K}$ and π induces $\overline{\pi} : U_H(G_0) \to G$. Since every homomorphism is determined by its effect on a generating set and $\Gamma = G_0 \coprod H$, $\overline{\pi}$ is unique. Our claims have been established.

Question 4.1. Is NZCT preserved in $U_H(G_0)$?

If so, $U_H(G_0)$ would be a model of $Th_\forall(H)$ and hence a model of $Th^0_\forall(H)$. Since universal sentences are preserved in subgroups, we would have G_0 a model of $Th^0_\forall(H)$.

We observe that

$$G_0 \cap Z(U_H(G_0)) \leq Z(G_0).$$

We further observe that $Z(G_0)$ coinciding with $G_0 \cap Z(U_H(G_0))$ is a necessary condition for Question 4.1 to have a positive answer. Recall we are taking G_0 nonabelian. Let g_1 and g_2 be noncommuting elements of G_0 and $g \in Z(G_0) \backslash (G_0 \cap Z(U_H(G_0)))$, then since g_1 and g_2 each commute with $g \notin Z(U_H(G_0))$, NZCT would be violated.

Question 4.2. Is $Z(G_0) = G_0 \cap Z(U_H(G_0))$?

5. Questions

Let $c \geq 2$ be an integer. Let $r = \max\{2, c-1\}$. Let s be any integer such that $s \geq r$. Let $G = F_s(\mathcal{N}_c)$. Then $F_\omega(\mathcal{N}_c)$ is discriminated by the family of retractions $F_\omega(\mathcal{N}_c) \to G$. It follows that the $F_n(\mathcal{N}_c)$ have the same universal theory relative to L_0 for all $n \geq r$ and that the $F_n(\mathcal{N}_c)$ have the same universal theory relative to $L_0[G]$ for all $n \geq s$ (See [GS 1]). We let S_c be the set of all universal sentences of L_0 true in $F_n(\mathcal{N}_c)$. Since $F_n(\mathcal{N}_c) \leq F_r(\mathcal{N}_c)$ for all $0 \leq n \leq r$ and universal sentences are preserved in subgroups, S_c is actually the set of all universal sentences of L_0 true in every free c-nilpotent group.

For each integer $n \geq 0$ we define $CT(n)$ to be the following universal sentence of L_0.

$$\begin{aligned}\forall x_1, x_2, x_3, w_1, ..., w_n \ ((([w_1, ..., w_n, x_2] &\neq 1) \wedge (([x_1, x_2] = 1) \wedge ([x_2, x_3] = 1)) \\ &\to ([x_1, x_3] = 1)).\end{aligned}$$

The interpretation of $CT(n)$ in any group G is that the relation of commutativity is transitive on $G \backslash Z_n(G)$ where $Z_n(G)$ is the n-th term of the upper central series of G.

Equivalently, it asserts that the centralizer of any element $g \in G \backslash Z_n(G)$ is abelian. It was shown in [2] that the free c-nilpotent groups satisfy $CT(c-1)$. Questions 5.1–5.4 below are also due to A.G. Myasnikov.

Question 5.1. Does $Q(H) \cup \{NZCT\}$ axiomatize $S_2 = Th^0_\forall(H)$?

A positive answer to Question 4.1 would imply a positive answer to Question 5.1.

Question 5.2. Let $s \geq 2$ and let $G = F_s(\mathcal{N}_2)$. Does $Q(G) \cup \text{diag}(G) \cup \{NZCT\}$ axiomatize $Th_\forall(G)$?

More generally -

Question 5.3. Let $c \geq 2$ and $r = \max\{2, c-1\}$. Let $G = F_n(\mathcal{N}_c)$. Does $Q^0(G) \cup \{CT(c-1)\}$ axiomatize S_c?

Question 5.4. Let $c \geq 2$ and $s \geq r = \max\{2, c-1\}$. Let $G = F_s(\mathcal{N}_c)$. Does $Q(G) \cup diag(G) \cup \{CT(c-1)\}$ axiomatize $Th_\forall(G)$?

Question 5.5. Let θ be an indeterminate over $\mathbb{Z}$. Must every finitely generated model of $Th_\forall(H)$ H-embed in $UT_3(\mathbb{Z}[\theta])$?

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