EQUATIONS IN VIRTUALLY CLASS 2 NILPOTENT GROUPS

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ABSTRACT. We give an algorithm that decides whether a single equation in a group that is virtually a class 2 nilpotent group with a virtually cyclic commutator subgroup, such as the Heisenberg group, admits a solution. This generalises the work of Duchin, Liang and Shapiro to finite extensions.

1. INTRODUCTION

Since the 1960s, many papers have discussed algorithms to decide whether or not equations in a variety of different classes of groups admit solutions. An *equation* with the variable set V in a group G has the form, $w = 1$, for some element $w \in G * F(V)$. A first major positive result in this area is due to Makanin, during the 1980s, when in a series of papers he proved that it is decidable whether a finite system of equations in a free group admits a solution [9–11]. Since then, Makanin's work has been extended to show the decidability of the satisfiability equations in hyperbolic groups, solvable Baumslag-Solitar groups, right-angled Artin groups and more [2,4,7].

Our primary focus in this paper will be single equations in virtually finitely generated class 2 nilpotent groups, where if $\mathcal{P}$ is a property of groups, we say a group is *virtually* $\mathcal{P}$ if it has a finite-index subgroup with $\mathcal{P}$. The *single equation problem* in a group G is the decision question as to whether there is an algorithm for G that takes as input an equation $w = 1$ in G and outputs whether or not $w = 1$ admits a solution. Duchin, Liang and Shapiro proved that the single equation problem in finitely generated class 2 nilpotent groups with a virtually cyclic commutator subgroup is decidable [5]. This is in contrast with the fact that the satisfiability of systems of equations in free nilpotent groups of class 2 is undecidable. The assumption that the commutator subgroup is virtually cyclic cannot be completely removed; Roman'kov gave an example of a finitely generated class 2 nilpotent group where it is undecidable whether equations of the form $[X_1, \ X_2] = g$, where g is a constant, admit solutions [15].

Our main result is to generalise Duchin, Liang and Shapiro's result to show that the single equation problem in virtually a finitely generated class 2 nilpotent group with a virtually cyclic commutator subgroup is decidable. This class includes the Heisenberg group and all higher Heisenberg groups.

Theorem 1.1. *The single equation problem in virtually a group that is class 2 nilpotent group with a virtually cyclic commutator subgroup is decidable.*

Roman'kov began the study of equations in nilpotent groups, when in 1977 he showed that in free nilpotent groups of class at least 9 and sufficiently large rank, it is undecidable whether finite systems of equations admit a solution [14]. Following this, Repin proved that there is a finitely presented nilpotent group such that the satisfiability of single equations with one variable are undecidable [12], and improved this by showing that there are such groups of nilpotency class 3 [13]. These results constrast with the fact that the conjugacy problem, which is a specific example of a one-variable equation, is decidable in all finitely generated nilpotent groups. Repin also showed that the satisfiability of single equations with one variable in non-abelian free nilpotent groups of class at least 10^{20} is undecidable [13].

In the positive direction, Repin proved that the satisfiability of single equations with one variable are decidable in any finitely generated class 2 nilpotent group [13]. In addition, Truss showed that the satisfiability of single equations in two variables in the free nilpotent group of class 2 and rank 2 (the Heisenberg group) are decidable [17].

We prove our main result using a similar method to the method used by Duchin, Liang and Shapiro [5]; by converting an equation in a virtually class 2 nilpotent group with a virtually cyclic commutator into an equivalent system of linear and quadratic equations and congruences in the ring of integers. We then show that the system obtained is of the same type as that obtained from an equation in a class 2 nilpotent group with a virtually cyclic commutator subgroup, and is thus decidable using the work of Duchin, Liang and Shapiro.

In Section 2, we define a group equation and solution, and give some background on nilpotent groups. In Section 3, we use the arguments of Duchin, Liang and Shapiro [5] to detail the reduction from a single equation in a class 2 nilpotent group to a system of equations in the ring of integers. We conclude in Section 4 by using this reduction to prove Theorem 1.1.

2. PRELIMINARIES

Notation 2.1. We introduce notation we will frequently use.
(1) If S is a subset of a group, we define $S^{\pm} = S \cup S^{-1}$. Moreover, if $a \in S$ and w is a word over $S^{\pm}$ we define $\mathrm{expsum}_a w$ to be the number of occurences of a in w minus the number of occurences of a^{-1} in w;
(2) For elements g and h of a group G, the commutator is defined by $[g,\ h] = g^{-1}h^{-1}gh$;
(3) If $x \in \mathbb{R}$, we will define the floor notation $\lfloor x \rfloor$ in a non-standard way:

$$\lfloor x \rfloor = \begin{cases} \max\{y \in \mathbb{Z} \mid y \leq x\} & x \geq 0 \\ \min\{y \in \mathbb{Z} \mid y \geq x\} & x < 0. \end{cases}$$

That is, we round towards zero.

Group equations. We start with the definition and some examples of equations in groups.

Definition 2.2. Let G be a finitely generated group, V be a finite set disjoint with G, and $F(V)$ be the free group on V. An *equation* in G is an identity $w = 1$, where $w \in G * F(V)$. A *solution* to $w = 1$ is a homomorphism $\phi \colon G * F(V) \to G$ that fixes elements of G, such that $\phi(w) = 1$. The elements of V are called the *variables* of the equation. A *system of equations* in G is a finite set of equations in G, and a *solution* to a system is a homomorphism that is a solution to every equation in the system.

We say two systems of equations in G are *equivalent* if they have the same set of solutions.

The *single equation problem* in G is the decidability question as to whether there is an algorithm that accepts as input an equation $w = 1$ in G, where the elements of G within $w = 1$ are represented by words over a finite generating set, and returns YES if $w = 1$ admits a solution and NO otherwise.

Remark 2.3. We will often write a solution to an equation in a group G as a tuple of elements $(g_1, \ldots, g_n)$, rather than a homomorphism. We can recover such a homomorphism ϕ from a tuple by setting $\phi(g) = g$ for each $g \in G$ and $\phi(X_i) = g_i$ for each X_i, and the action of ϕ on the remaining elements is now determined as it is a homomorphism.

Example 2.4. Deciding whether an equation in the group $\mathbb{Z}$ admits a solution reduces to solving a linear equation in integers. For example, using the free generator a for $\mathbb{Z}$,

$$X^2 a^3 Y^2 a^{-3} Y^{-1} a = 1$$

is an equation, which we can rewrite using additive notation as

$$2X + 3 + 2Y - 3 - Y + 1 = 0.$$

We can use the fact that $\mathbb{Z}$ is abelian to show that this is equivalent to $2X + Y + 1 = 0$, which is just a linear equation in integers, and elementary linear algebra can be used to decide if it admits a solution (and 'construct' the set of solutions). In this case, the equation does admit solutions, and the set of solutions is

$$\{(x, \ -2x - 1) \mid x \in \mathbb{Z}\}.$$

Nilpotent groups. Below we give the definition of a nilpotent group, along with an elementary lemma about commutators we will use later on.

Definition 2.5. Let G be a group. Define $\gamma_i(G)$ for all $i \in \mathbb{Z}_{\geq 0}$ inductively as follows.

$$\gamma_0(G) = G$$
$$\gamma_i(G) = [G, \ \gamma_{i-1}(G)] \text{ for } i \geq 1.$$

The subnormal series $(\gamma_i(G))$ is called the *lower central series* of G. We call G *nilpotent* of *class c* if $\gamma_c(G)$ is trivial.

Lemma 2.6 ([8], Lemma 2.3). *Let G be a class 2 nilpotent group, and $g, \ h \in G$. Then*
1. $[g^{-1}, \ h^{-1}] = [g, \ h]$,
2. $[g^{-1}, \ h] = [g, \ h]^{-1}$.

We now introduce the normal form we will be using for class 2 nilpotent groups. This is used in [5], and we include the proof of uniqueness and existence for completeness.

The following lemma is used to define the Mal'cev generating set and normal form.

Lemma 2.7. *Let G be a class 2 nilpotent group with a virtually cyclic commutator subgroup. Then G has a generating set*

$$\{a_1, \ldots, a_n, b_1, \ldots, b_r, c, d_1, \ldots, d_t\},$$

where n, r, $t \in \mathbb{Z}_{>0}$, such that the d_is have finite order, c and the d_is are central, for each b_i, there exists $l_i \in \mathbb{Z}_{>0}$, such that $b_i^{l_i} \in [G, G]$, and $[G, G] = \langle c, d_1, \ldots, d_t\rangle$.

Moreover, every element of G can be expressed uniquely as an element of the set

$$\{a_1^{i_1} \cdots a_n^{i_n} b_1^{j_1} \cdots b_r^{j_r} c^p d_1^{q_1} \cdots d_t^{q_t} \mid i_1, \ldots, i_n, p \in \mathbb{Z}, \tag{2.1}$$
$$j_x \in \{0, \ldots, l_x - 1\}, q_x \in \{0, \ldots, k_x - 1\} \text{ for each } x\}.$$

Proof. Using the fundamental theorem for finitely generated abelian groups and the fact that $[G, G]$ is virtually cyclic, the short exact sequence $\{1\} \to [G, G] \to G \to {}^{G}\!/_{[G, G]} \to \{1\}$ becomes

$$\{1\} \longrightarrow \mathbb{Z} \oplus (\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_t}) \longrightarrow G \longrightarrow \mathbb{Z}^n \oplus (\mathbb{Z}_{l_1} \oplus \cdots \oplus \mathbb{Z}_{l_r}) \longrightarrow \{1\},$$

where $n \in \mathbb{Z}_{>0}$ and r, $t \in \mathbb{Z}_{\geq 0}$. Let $a_1, \ldots, a_n$ be lifts in G of standard generators for $\mathbb{Z}^n$, $b_1, \ldots, b_r$ be lifts of generators of $\mathbb{Z}_{l_1}, \ldots, \mathbb{Z}_{l_r}$, respectively. Let c be a generator for $\mathbb{Z}$, and $d_1, \ldots, d_t$ be generators of $\mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_t}$, respectively. Then using our short exact sequence, it follows that $\{a_1, \ldots, a_n, b_1, \ldots, b_r, c, d_1, \ldots, d_t\}$ generates G. We have that $d_i^{k_i} = 1$, for all i. As $\{c, d_1, \ldots, d_t\}$ generates $[G, G]$, the and we have shown that the generating set exists.

We now turn our attention to the normal form, showing existence and uniqueness.

Existence: Let $g \in G$, and w be a word over our generating set that represents g. As c and all d_is are central, we can push them to the back of w, and into the desired order. As $[a_i, a_j]$, $[b_i, b_j]$, and $[a_i, b_j]$ can be written as expressions using c the and d_is, we have that reordering the a_is and b_is to the desired form simply creates expressions using cs and the d_is, which can then be pushed to the back of w, and into the stated order. Let $i \in \{1, \ldots, r\}$. By definition, $[G, G]b_i^{l_i} = [G, G]$, so we can reduce b_i modulo l_i by creating an expression over c and the d_is, which, again, can be pushed to the back and into the desired form. Since the d_is have finite order, we can reduce their exponents modulo these orders.

Uniqueness: Let $i_1, \ldots, i_{n+r+1+t} \in \mathbb{Z}$ and $j_1, \ldots, j_{n+r+1+t} \in \mathbb{Z}$ be such that

$$u = a_1^{i_1} \cdots a_n^{i_n} b_1^{i_{n+1}} \cdots b_r^{i_{n+r}} c^{i_{n+r+1}} d_1^{i_{n+r+1+1}} \cdots d_t^{i_{n+r+1+t}}$$

and

$$v = a_1^{j_1} \cdots a_n^{j_n} b_1^{j_{n+1}} \cdots b_r^{j_{n+r}} c^{j_{n+r+1}} d_1^{j_{n+r+1+1}} \cdots d_t^{j_{n+r+1+t}}$$

and expressions in the normal form stated in the lemma. Suppose $u =_G v$. Then u and v have the same image in the quotient of G by $[G, G]$, and so $a_1^{i_1} \cdots a_n^{i_n} b_1^{i_{n+1}} \cdots b_r^{i_{n+r}} =_{G/[G, G]} a_1^{j_1} \cdots a_n^{j_n} b_1^{j_{n+1}} \cdots b_r^{j_{n+r}}$. As these words are in the standard normal form for these finitely generated abelian groups, it follows that $i_x = j_x$ for all $x \in \{1, \ldots, n + r\}$. Thus

$$c^{i_{n+r+1}} d_1^{i_{n+r+1+1}} \cdots d_t^{i_{n+r+1+t}} =_G c^{j_{n+r+1}} d_1^{j_{n+r+1+1}} \cdots d_t^{j_{n+r+1+t}}.$$

As $[G, G]$ is abelian and these words are in the standard normal form for $[G, G]$, it follows that all of the exponents in these expressions must be equal; that is $i_x = j_x$ for all $x \in \{n + r + 1, \ldots, n + r + 1 + t\}$. We have now shown that u and v are equal as words, as required.

Definition 2.8. A generating set defined as in Lemma 2.7 is called a *Mal'cev generating set*, and the normal form defined in Lemma 2.7 is called the *Mal'cev normal form*.

As we have seen in the proofs of the previous lemma, one can manipulate words in class 2 nilpotent groups with a virtually cyclic commutator subgroup by pushing a_is past b_is and paying a 'cost' in c and the d_is. Quantifying the 'cost' for each such move will be necessary to convert a given equation in a class 2 nilpotent group into a system of equations in the ring of integers, with the 'cost' appearing as constants in this system.

Notation 2.9. We define a number of values for a group G with the Mal'cev generating set
$$\{a_1, \ldots, a_n, b_1, \ldots, b_r, c, d_1, \ldots, d_t\},$$
where again, l_i is minimal (and exists) such that $b_i^{l_i} \in [G, G]$ and the order of d_i is k_i.

(1) From Lemma 2.7, we have that $[a_i, a_j], [b_i, b_j], [a_i, b_j] \in \{c^p d_1^{q_1} \cdots d_t^{q_t} \mid p, q_1, \ldots, q_t \in \mathbb{Z}\}$, for all i, j, with $i < j$ in the first two expressions. For all such i and j, $k \in \{1, \ldots, s\}$, and $l \in \{1, \ldots, t\}$, we can therefore define $\alpha_{ijl}, \beta_{ijl}$ and γ_{ijl} to be the unique integers satisfying the following normal form expressions in $[G, G]$:
$$[a_j, a_i] = c^{\alpha_{ij0}} d_1^{\alpha_{ij1}} \cdots d_t^{\alpha_{ijt}}, \quad (i < j)$$
$$[b_j, b_i] = c^{\beta_{ij0}} d_1^{\beta_{ij1}} \cdots d_t^{\beta_{ijt}}, \quad (i < j)$$
$$[a_i, b_j] = c^{\gamma_{ij0}} d_1^{\gamma_{ij1}} \cdots d_t^{\gamma_{ijt}}.$$

(2) Since $b_i^{l_i} \in [G, G]$, we can define η_{il} for all $i \in \{1, \ldots, r\}$, and $l \in \{0, \ldots, t\}$, to be the unique integers such that
$$b_i^{l_i} = c^{\eta_{i0}} d_1^{\eta_{i1}} \cdots d_t^{\eta_{it}}.$$

3. Transforming equations in nilpotent groups into equations in integers

This section aims to prove Lemma 3.6; that is, a single equation $\mathcal{E}$ in a class 2 nilpotent group with a virtually cyclic commutator subgroup is equivalent to a system $S_{\mathcal{E}}$ over $\mathbb{Z}$ of (1) linear equations and congruences, (2) a single quadratic equation and (3) quadratic congruences, where the quadratic equations and congruences can also contain 'floor' terms.

The idea of the proof is to replace each variable in $\mathcal{E}$ with a word representing a potential solution, and then convert this new word into Mal'cev normal form. The linear equations in $S_{\mathcal{E}}$ occur as the solution to the exponent of each generator a_i being set to 0, and the linear congruences, quadratic equation and quadratic congruences occur when the same is done for the b_is, c and the d_is, respectively.

We begin with an example of this process.

Example 3.1. Let G be the class 2 nilpotent group with the presentation
$$\langle a_1, a_2, b, c, d \mid c = [a_1, a_2], d = [a_1, b] = [a_2, b], b^2 = c, d^2 = 1,$$
$$[a_1, c] = [a_1, d] = [a_2, c] = [a_2, d] = [b, c] = [b, d] = 1\rangle.$$
Consider the equation
$$Xba_1cXa_2c^{-3}a_1X = 1 \tag{3.1}$$

We first transform the constants in this equation into Mal'cev normal form, push all the commutators to the right, and then use the relation $d^2 = 1$ to obtain

$$Xa_1bXa_1a_2Xc^{-3}d = 1. \tag{3.2}$$

We set $X = a_1^{X_1}a_2^{X_2}b^{X_3}c^{X_4}d^{X_5}$ using our Mal'cev normal form, for new variables $X_1, \ldots, X_5$ over $\mathbb{Z}$. Plugging this into (3.2) gives

$$a_1^{X_1}a_2^{X_2}b^{X_3}c^{X_4}d^{X_5}a_1ba_1^{X_1}a_2^{X_2}b^{X_3}c^{X_4}d^{X_5}a_1a_2a_1^{X_1}a_2^{X_2}b^{X_3}c^{X_4}d^{X_5}c^{-3}d = 1. \tag{3.3}$$

We can the transform this into Mal'cev normal form, to (first) obtain

$$a_1^{3X_1+2}a_2^{3X_2+1}b^{3X_3+1}c^{3X_4+X_1(1+X_2+X_2)+X_1X_2-3} \tag{3.4}$$
$$d^{3X_5+X_2(X_3+1+X_3)+X_1(X_3+1+X_3)+(X_3+1+X_3)+(X_3+1+X_3)+X_2(1+X_3)+X_1(1+X_3)+X_3+2+1} = 1.$$

Simplifying this gives

$$a_1^{3X_1+2}a_2^{3X_2+1}b^{3X_3+1}c^{3X_1X_2+X_1+3X_4-3}d^{3X_1X_3+3X_2X_3+4X_1+4X_2+5X_3+3X_5+5} = 1. \tag{3.5}$$

Using the relations $b^2 = c$ and $d^2 = 1$, we can conclude

$$a_1^{3X_1+2}a_2^{3X_2+1}b^{(X_3+1)\bmod 2}c^{3X_1X_2+X_1+3X_4-3+\left\lfloor\frac{X_3+1}{2}\right\rfloor+X_3}d^{(X_1X_3+X_2X_3+X_3+X_5+1)\bmod 2} = 1. \tag{3.6}$$

This results in the following system of equations over (the ring) $\mathbb{Z}$

$$3X_1 + 2 = 0 \tag{3.7}$$
$$3X_2 + 1 = 0$$
$$X_3 + 1 \equiv 0 \pmod 2$$
$$3X_1X_2 + X_1 + X_3 + \left\lfloor\frac{X_3+1}{2}\right\rfloor + 3X_4 - 3 = 0$$
$$X_1X_3 + X_2X_3 + X_3 + X_5 + 1 \equiv 0 \pmod 2.$$

As $3X_1 + 2 = 0$ admits no integer solutions, we can conclude that our equation (3.1) does not admit a solution.

Notation 3.2. Let G be a class 2 nilpotent group, $X_1, \ldots, X_M$ be variables, where $M \in \mathbb{Z}_{>0}$. Let $N \in \mathbb{Z}_{>0}$, $\epsilon_1, \ldots \epsilon_N \in \{-1, 1\}$, and

$$\omega_1 X_{p_1}^{\epsilon_1} \cdots \omega_N X_{p_N}^{\epsilon_N} = 1 \tag{3.8}$$

be an equation over G, where $\omega_1, \ldots, \omega_N$ are words in Mal'cev normal form, over a Mal'cev generating set for G, as constructed in Lemma 2.7 and $p_1, \ldots, p_M \in \{1, \ldots, M\}$. We will also use the notation introduced in Lemma 2.7 for the generators. We will use $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N$ to be a potential solution. We define a number of values based on the ω_zs and $\boldsymbol{\nu}_z$s. To make it clearer, the potential solution is shown in bold.

For each Mal'cev generator a, we will define $\boldsymbol{\nu}_{z,a}$ and $\omega_{z,a}$ by:

$$\boldsymbol{\nu}_{z,a} = \text{expsum}_a(\boldsymbol{\nu}_z), \qquad \omega_{z,a} = \text{expsum}_a(\omega_z).$$

By convention we will often use $d_0 = c$ and take $\equiv_{k_0}$ to be equality (since c is infinite order and k_0 is being used to represent the order of $d_0 = c$, this equality modulo k_0 is true equality).

This lemma transforms the equation that we obtained from (3.8) to obtain a system of linear and quadratic equations and congruences over the integers. We do this by first transforming the equation with the potential solution subbed in into Mal'cev normal form. This corresponds to moving from (3.4) to (3.6) in Example 3.1. Following that, we equate all of the exponents in this word to zero, given our system, which is done to obtain (3.7) in Example 3.1, respectively. The capital Latin alphabet characters are constants derived from the constants of the equation, and the group's structure. Recall that the $\boldsymbol{\nu}_{z,a}$s represent variables over $\mathbb{Z}$ (see Notation 3.2). For $i \in \{1, -1\}$ will use $\delta_i = 1$ if $i = -1$ and $\delta_i = 0$ otherwise.

Lemma 3.3. *The words $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_M$ form a solution to (3.8) in a class 2 nilpotent group with a virtually cyclic commutator subgroup, if and only if the following equations and congruences hold:*

$$A_m + \sum_{z=1}^{N} \epsilon_z \boldsymbol{\nu}_{p_z,a_m} = 0, \tag{3.9}$$

$$B_m + \sum_{z=1}^{N} \epsilon_z \boldsymbol{\nu}_{p_z,b_m} \equiv_{l_m} 0, \tag{3.10}$$

$$D_m + \sum_{z=1}^{N} \epsilon_z \boldsymbol{\nu}_{p_z,d_m} - \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{j=1}^{n} \epsilon_u \boldsymbol{\nu}_{p_u,a_j} K_{mzj} + \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{j=1}^{r} \epsilon_u \boldsymbol{\nu}_{p_u,b_j} L_{mzj} - \sum_{\substack{z,u=1 \\ u\leq z}}^{N} \sum_{i=1}^{n} \epsilon_t \boldsymbol{\nu}_{p_t,a_i}$$

$$\tag{3.11}$$

$$- J_{mui} - \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{n} \epsilon_z \epsilon_u \boldsymbol{\nu}_{p_z,a_i} \boldsymbol{\nu}_{p_u,a_j} \alpha_{ijm} - \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{i=1}^{n} \sum_{j=1}^{r} \epsilon_z \epsilon_u \boldsymbol{\nu}_{p_z,a_i} \boldsymbol{\nu}_{p_u,b_j} \gamma_{ijm}$$

$$- \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{j=1}^{r} \boldsymbol{\nu}_{p_u,b_j} M_{mzj} - \sum_{\substack{z,u=1 \\ u\leq z}}^{N} \sum_{i=1}^{r} \epsilon_z \boldsymbol{\nu}_{p_z,b_i} - O_{mui} - \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{r} \epsilon_z \epsilon_u \boldsymbol{\nu}_{p_z,b_i} \boldsymbol{\nu}_{p_u,b_j} \beta_{ijm}$$

$$- \sum_{z=1}^{N} \sum_{i=1}^{r} \eta_{im} \left\lfloor \frac{\omega_{z,b_i} + \epsilon_z \boldsymbol{\nu}_{p_z,b_i}}{l_i} \right\rfloor$$

$$- \sum_{z=1}^{N} \delta_{\epsilon_z} \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} \alpha_{ijm} \boldsymbol{\nu}_{p_z,a_i} \boldsymbol{\nu}_{p_z,a_j} + \sum_{i=1}^{n} \sum_{j=1}^{r} \gamma_{ijm} \boldsymbol{\nu}_{p_z,a_i} \boldsymbol{\nu}_{p_z,b_j} + \sum_{\substack{i,j=1 \\ i<j}}^{r} \beta_{ijm} \boldsymbol{\nu}_{p_z,b_i} \boldsymbol{\nu}_{p_z,b_j} \right) \equiv_{k_m} 0$$

where for all m

$$A_m = \sum_{z=1}^{N} \omega_{z,a_m}, \qquad\qquad B_m = \sum_{z=1}^{N} \omega_{z,b_m},$$

$$J_{mui} = \sum_{\substack{j=1 \\ i<j}}^{n} \omega_{u,a_j}\alpha_{ijm} + \sum_{j=1}^{r} \omega_{u,b_j}\gamma_{ijm}, \qquad\qquad K_{mzj} = \sum_{\substack{i=1 \\ i<j}}^{n} \omega_{z,a_i}\alpha_{ijm},$$

$$L_{mzj} = \sum_{i=1}^{n} \omega_{z,a_i}\gamma_{ijm}, \qquad\qquad M_{mzj} = \sum_{\substack{i=1 \\ i<j}}^{r} \omega_{z,b_i}\beta_{ijm},$$

$$O_{mui} = \sum_{\substack{j=1 \\ i<j}}^{r} \omega_{u,b_j}\beta_{ijm},$$

$$D_m = \sum_{z=1}^{N} \omega_{z,d_m}$$
$$- \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{n} \omega_{z,a_i}\alpha_{ijm}\omega_{u,a_j} - \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{z,a_i}\gamma_{ijm}\omega_{u,b_j} - \sum_{\substack{z,u=1 \\ u<z}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{r} \omega_{z,b_i}\beta_{ijm}\omega_{u,b_j}.$$

Proof. Let $w = \omega_1 \boldsymbol{\nu}_{p_1}^{\epsilon_1} \cdots \omega_M \boldsymbol{\nu}_{p_M}^{\epsilon_M}$ by the left-hand side of the equation (3.8), with the potential solution $(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N)$ plugged in. By pushing the d_ms (recall that $c = d_0$) to the end of w, we have that w now comprises $2N$ words over $\{a_1, \ldots, b_r\}^{\pm}$ in Mal'cev normal form followed by an expression of d_ms. We will now convert w into Mal'cev normal form, in order to compare the exponents of the generators of this normal form version for w to 0. From now on, whenever we modify w, we will continue to use the fact that the d_ms are central to push them to the right.

Using Notation 2.9, if $i < j$, then $a_j a_i = a_i a_j [a_i,\ a_j]^{-1} = a_i a_j d_0^{-\alpha_{ij0}} d_1^{-\alpha_{ij1}} \cdots d_t^{-\alpha_{ijt}}$ and $b_j b_i = b_i b_j d_0^{-\beta_{ij0}} d_1^{-\beta_{ij1}} \cdots d_t^{-\beta_{ijt}}$. Similarly, for any i and j, $b_j a_i = a_i b_j d_0^{-\gamma_{ij0}} d_1^{-\gamma_{ij1}} \cdots d_t^{-\gamma_{ijt}}$. We will use this to reorder all of the subwords $(a_1^{\boldsymbol{\nu}_{p_z,a_1}} \cdots a_n^{\boldsymbol{\nu}_{p_z,a_n}} b_1^{\boldsymbol{\nu}_{p_z,b_1}} \cdots b_r^{\boldsymbol{\nu}_{p_z,b_r}})^{\epsilon_z}$ into a word within $(a_1^{\pm})^* \cdots (a_n^{\pm})^*(b_1^{\pm})^* \cdots (b_r^{\pm})^*(d_0^{\pm})^*(d_1^{\pm})^* \cdots (d_t^{\pm})^*$, subject to 'creating' some additional commutators, which are then pushed to the right. Note that if $\epsilon_z = 1$, then the word is already in the desired form, so consider when $\epsilon_z = -1$. Let $u = \boldsymbol{\nu}_z^{\epsilon_z}$ be such a subword (that is, $\epsilon_z = -1$). Then

$$u = b_r^{-\boldsymbol{\nu}_{p_z,b_r}} \cdots b_1^{-\boldsymbol{\nu}_{p_z,b_1}} a_n^{-\boldsymbol{\nu}_{p_z,a_n}} \cdots a_1^{-\boldsymbol{\nu}_{p_z,a_1}}.$$

We will start at the right, and push terms to the left. We have that the a_1s will have to be pushed past everything (except each other), the a_2s will need to be pushed past everything except the a_1s, and so on up to the b_{r-1}s, which will only need to be pushed past the b_rs, and the b_rs which will not need to be pushed past anything, as they will now be in the

correct place. Thus

$$
\begin{aligned}
u = a_1^{-\boldsymbol{\nu}_{p_z,a_1}} &\cdots a_n^{-\boldsymbol{\nu}_{p_z,a_n}} b_1^{-\boldsymbol{\nu}_{p_z,b_1}} \cdots b_r^{-\boldsymbol{\nu}_{p_z,b_r}} \\
&\prod_{m=0}^{t} d_m^{\displaystyle -\sum_{\substack{i,j=1\\i<j}}^{n} \alpha_{ijm}\boldsymbol{\nu}_{p_z,a_i}\boldsymbol{\nu}_{p_z,a_j} - \sum_{i=1}^{n}\sum_{j=1}^{r} \gamma_{ijm}\boldsymbol{\nu}_{p_z,a_i}\boldsymbol{\nu}_{p_z,b_j} - \sum_{\substack{i,j=1\\i<j}}^{r} \beta_{ijm}\boldsymbol{\nu}_{p_z,b_i}\boldsymbol{\nu}_{p_z,b_j}} .
\end{aligned}
$$

Now consider the general case for $u = \boldsymbol{\nu}_z^{\epsilon_z}$, with $\epsilon_z \in \{-1,\ 1\}$. We have

$$
\begin{aligned}
u = a_1^{\epsilon_z\boldsymbol{\nu}_{p_z,a_1}} &\cdots a_n^{\epsilon_z\boldsymbol{\nu}_{p_z,a_n}} b_1^{\epsilon_z\boldsymbol{\nu}_{p_z,b_1}} \cdots b_r^{\epsilon_z\boldsymbol{\nu}_{p_z,b_r}} \\
&\prod_{m=0}^{t} d_m^{\displaystyle -\delta_{\epsilon_z}\left(\sum_{\substack{i,j=1\\i<j}}^{n} \alpha_{ijm}\boldsymbol{\nu}_{p_z,a_i}\boldsymbol{\nu}_{p_z,a_j} + \sum_{i=1}^{n}\sum_{j=1}^{r} \gamma_{ijm}\boldsymbol{\nu}_{p_z,a_i}\boldsymbol{\nu}_{p_z,b_j} + \sum_{\substack{i,j=1\\i<j}}^{r} \beta_{ijm}\boldsymbol{\nu}_{p_z,b_i}\boldsymbol{\nu}_{p_z,b_j} \right)} .
\end{aligned}
$$

We now push all a_is to the left, whilst calculating the cost in d_ms. For ω_1 there is nothing to do. For $\boldsymbol{\nu}_1$, we have $\boldsymbol{\nu}_{1,a_i}$ a_is, and we must move each of these past ω_{1,a_j} a_js (where $j > i$), and past ω_{1,b_j} b_js, (where j is arbitrary). So moving the a_is to the left (provided all lower indexed a_is have already been moved) will increase the number of d_ms by

$$
- \sum_{\substack{i,j=1\\i<j}}^{n} \epsilon_1\boldsymbol{\nu}_{p_1,a_i}\omega_{1,a_j}\alpha_{ijm} - \sum_{i=1}^{n}\sum_{j=1}^{r} \epsilon_1\boldsymbol{\nu}_{p_1,a_i}\omega_{1,b_j}\gamma_{ijm},
$$

Doing the same for the a_is in ω_2, we will now have to push them past the a_js and b_js in ω_1 and $\boldsymbol{\nu}_1$, so this will increase the number of d_ms by

$$
- \sum_{\substack{i,j=1\\i<j}}^{n} \omega_{2,a_i}\alpha_{ijm}(\omega_{1,a_j} + \epsilon_1\boldsymbol{\nu}_{p_1,a_j}) - \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{2,a_i}\gamma_{ijm}(\omega_{1,b_j} + \epsilon_1\boldsymbol{\nu}_{p_1,b_j}),
$$

respectively. Proceeding in this manner for the remaining ω_ts and $\boldsymbol{\nu}_t$s gives the total increase of the d_ms as

$$
- \sum_{\substack{t,u=1\\u<t}}^{N} \left(\sum_{\substack{i,j=1\\i<j}}^{n} \omega_{t,a_i}\alpha_{ijm}(\omega_{u,a_j} + \epsilon_u\boldsymbol{\nu}_{p_u,a_j}) + \sum_{i=1}^{n}\sum_{j=1}^{r} \omega_{t,a_i}\gamma_{ijm}(\omega_{u,b_j} + \epsilon_u\boldsymbol{\nu}_{p_u,b_j}) \right)
$$

$$
- \sum_{\substack{t,u=1\\u\leq t}}^{N} \left(\sum_{\substack{i,j=1\\i<j}}^{n} \epsilon_t\boldsymbol{\nu}_{p_t,a_i}\omega_{u,a_j}\alpha_{ijm} + \sum_{i=1}^{n}\sum_{j=1}^{r} \epsilon_t\boldsymbol{\nu}_{p_t,a_i}\omega_{u,b_j}\gamma_{ijm} \right)
$$

$$
- \sum_{\substack{t,u=1\\u<t}}^{N} \left(\sum_{\substack{i,j=1\\i<j}}^{n} \epsilon_t\epsilon_u\boldsymbol{\nu}_{p_t,a_i}\boldsymbol{\nu}_{p_u,a_j}\alpha_{ijm} + \sum_{i=1}^{n}\sum_{j=1}^{r} \epsilon_t\epsilon_u\boldsymbol{\nu}_{p_t,a_i}\boldsymbol{\nu}_{p_u,b_j}\gamma_{ijm} \right).
$$

This occurs in (3.4) and (3.5) in Example 3.1. We will now reorder the b_is, which occurs in (3.4) and (3.5) in Example 3.1. Again, those in ω_1 are already in position, and pushing those in $\boldsymbol{\nu}_1$ into place increases the number of d_ms by

$$-\sum_{\substack{i,j=1 \\ i<j}}^{r} \epsilon_1 \boldsymbol{\nu}_{p_1,b_i} \omega_{1,b_j} \beta_{ijm}.$$

Doing the same for ω_2 increases the number by

$$-\sum_{\substack{i,j=1 \\ i<j}}^{r} \omega_{2,b_i} \beta_{ijm} (\omega_{1,b_j} + \epsilon_1 \boldsymbol{\nu}_{p_1,b_j}).$$

Doing the same for all ω_zs and $\boldsymbol{\nu}_{p_z}$s increases the exponent sum of the d_ms by

$$-\sum_{\substack{t,u=1 \\ u<t}}^{N} \left(\sum_{\substack{i,j=1 \\ i<j}}^{r} \omega_{t,b_i} \beta_{ijm} (\omega_{u,b_j} + \epsilon_u \boldsymbol{\nu}_{p_u,b_j}) \right) - \sum_{\substack{t,u=1 \\ u\leq t}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{r} \epsilon_t \boldsymbol{\nu}_{p_t,b_i} \omega_{u,b_j} \beta_{ijm} - \sum_{\substack{t,u=1 \\ u<t}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{r} \epsilon_t \epsilon_u \boldsymbol{\nu}_{p_t,b_i} \boldsymbol{\nu}_{p_u,b_j} \beta_{ijm}.$$

It remains to reduce the b_is with respect to their modularities, as is done in (3.6) in Example 3.1. We have that doing so increases the number of d_ms by

$$-\sum_{t=1}^{N} \sum_{i=1}^{r} \eta_{im} \left\lfloor \frac{\omega_{t,b_i} + \epsilon_t \boldsymbol{\nu}_{p_t,b_i}}{l_i} \right\rfloor,$$

respectively. Recall that our floor terms round towards zero. We have now converted w to normal form. So the normal form version of w is trivial if and only if all of the exponents of the a_is, b_is, d_is in its normal form are equal to 0. That is, for all valid m, the following system of equations hold. As each of the following equations is computed by setting the exponent of a generator to 0, we give the generator responsible for each equation in brackets next to the equation. This corresponds to going from (3.6) to (3.7) in Example 3.1. We given in brackets at the left of each equation the generator that is being equated to zero to

obtain this equation. Recall again we are using d_0 to represent c.

$$(a_m) \quad \sum_{t=1}^{N} \omega_{t,a_m} + \sum_{t=1}^{N} \epsilon_t \boldsymbol{\nu}_{p_t,a_m} = 0,$$

$$(b_m) \quad \sum_{t=1}^{N} \omega_{t,b_m} + \sum_{t=1}^{N} \epsilon_t \boldsymbol{\nu}_{p_t,b_m} \equiv_{l_m} 0,$$

$$(d_m) \quad \sum_{t=1}^{N} \omega_{t,d_m} + \sum_{t=1}^{N} \epsilon_t \boldsymbol{\nu}_{p_t,d_m}$$

$$- \sum_{\substack{t,u=1 \\ u<t}}^{N} \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} \omega_{t,a_i} \alpha_{ijm} (\omega_{u,a_j} + \epsilon_u \boldsymbol{\nu}_{p_u,a_j}) + \sum_{i=1}^{n} \sum_{j=1}^{r} \omega_{t,a_i} \gamma_{ijm} (\omega_{u,b_j} + \epsilon_u \boldsymbol{\nu}_{p_u,b_j}) \right)$$

$$- \sum_{\substack{t,u=1 \\ u\leq t}}^{N} \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} \epsilon_t \boldsymbol{\nu}_{p_t,a_i} \omega_{u,a_j} \alpha_{ijm} + \sum_{i=1}^{n} \sum_{j=1}^{r} \epsilon_t \boldsymbol{\nu}_{p_t,a_i} \omega_{u,b_j} \gamma_{ijm} \right)$$

$$- \sum_{\substack{t,u=1 \\ u<t}}^{N} \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} \epsilon_t \epsilon_u \boldsymbol{\nu}_{p_t,a_i} \boldsymbol{\nu}_{p_u,a_j} \alpha_{ijm} + \sum_{i=1}^{n} \sum_{j=1}^{r} \epsilon_t \epsilon_u \boldsymbol{\nu}_{p_t,a_i} \boldsymbol{\nu}_{p_u,b_j} \gamma_{ijm} \right)$$

$$- \sum_{\substack{t,u=1 \\ u<t}}^{N} \left(\sum_{\substack{i,j=1 \\ i<j}}^{r} \omega_{t,b_i} \beta_{ijm} (\omega_{u,b_j} + \epsilon_u \boldsymbol{\nu}_{p_u,b_j}) \right) - \sum_{\substack{t,u=1 \\ u\leq t}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{r} \epsilon_t \boldsymbol{\nu}_{p_t,b_i} \omega_{u,b_j} \beta_{ijm}$$

$$- \sum_{\substack{t,u=1 \\ u<t}}^{N} \sum_{\substack{i,j=1 \\ i<j}}^{r} \epsilon_t \epsilon_u \boldsymbol{\nu}_{p_t,b_i} \boldsymbol{\nu}_{p_u,b_j} \beta_{ijm} - \sum_{t=1}^{N} \sum_{i=1}^{r} \eta_{im} \left\lfloor \frac{\omega_{t,b_i} + \epsilon_t \boldsymbol{\nu}_{p_t,b_i}}{l_i} \right\rfloor$$

$$- \sum_{z=1}^{N} \delta_{\epsilon_z} \left(\sum_{\substack{i,j=1 \\ i<j}}^{n} \alpha_{ijm} \boldsymbol{\nu}_{p_z,a_i} \boldsymbol{\nu}_{p_z,a_j} + \sum_{i=1}^{n} \sum_{j=1}^{r} \gamma_{ijm} \boldsymbol{\nu}_{p_z,a_i} \boldsymbol{\nu}_{p_z,b_j} + \sum_{\substack{i,j=1 \\ i<j}}^{r} \beta_{ijm} \boldsymbol{\nu}_{p_z,b_i} \boldsymbol{\nu}_{p_z,b_j} \right) \equiv_{k_m} 0$$

Replacing constants in these equations with the constants stated in the lemma completes the proof.

We use the following definitions to restate Lemma 3.3 in an easier format.

Definition 3.4. A *quadratic function* from $\mathbb{Z}^n$ to $\mathbb{Z}$, where $n \in \mathbb{Z}_{>0}$, is a function $f \colon \mathbb{Z}^n \to \mathbb{Z}$ such that there exist $a_{ij} \in \mathbb{Z}$ for each $i, j \in \{1, \ldots, n\}$ and $b_1, \ldots, b_n, c \in \mathbb{Z}$, such that for all $(x_1, \ldots, x_n) \in \mathbb{Z}^n$,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{m} b_i x_i + c.$$

A *linear function* from $\mathbb{Z}^n \to \mathbb{Z}$ is a function $f \colon \mathbb{Z}^n \to \mathbb{Z}$, such that there exist $b_1, \ldots, b_n, c \in \mathbb{Z}$, such that for all $(x_1, \ldots, x_n) \in \mathbb{Z}^n$,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^m b_i x_i + c.$$

Definition 3.5. Let $w = 1$ be an equation in a class 2 nilpotent group. The system of equations in the ring of integers obtained by equating the exponents in the Mal'cev normal form for the equation to zero (that is, the system obtained in Lemma 3.3) is called the $\mathbb{Z}$-*system* of $w = 1$.

We now restate Lemma 3.3 up to grouping constants, and renaming constants and variables.

Lemma 3.6. *The $\mathbb{Z}$-system of a single equation $w = 1$ in a class 2 nilpotent group G with a virtually cyclic commutator subgroup is equivalent to a finite system of linear equations and congruences in $\mathbb{Z}$, together with the following equations and congruences for finitely many k:*

$$\sum_{i=1}^n -\alpha_i Y_i + f(X_1, \ldots, X_m) + \sum_{i=1}^m \epsilon_i \left\lfloor \frac{\beta_i X_i + \kappa_i}{\gamma_i} \right\rfloor = 0, \tag{3.12}$$

$$g_k(X_1, \ldots, X_m) + \sum_{i=1}^m \zeta_{ki} \left\lfloor \frac{\mu_{ki} X_i + \chi_{ki}}{\lambda_{ki}} \right\rfloor \equiv 0 \mod \delta_k, \tag{3.13}$$

where the values with Greek alphabet names are all constants computable from the class 2 nilpotent group and the single equation, $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are variables, and the f and the g_ks are quadratic functions.

Lemma 3.7 follows with a little work from the result of Siegel that the satisfiability of single quadratic equations in the ring of integers is decidable [16]. We refer the reader to [5] for the proof.

Lemma 3.7 ([5], Section 2.2)**.** *It is decidable whether a system of equations of the form stated in Lemma 3.6 admits a solution.*

4. Equations in virtually nilpotent groups

Within this section, we look at how equations behave when passing to a finite index overgroup. From [5], we know that the single equation problem is decidable in any class 2 nilpotent group with a virtually cyclic commutator subgroup. Doing so requires an understanding of how automorphisms of such groups behave. We start by investigating these.

Proposition 4.1. *Let G be a class 2 nilpotent group with a virtually cyclic commutator subgroup. Let $\{a_1, \ldots, a_n, b_1, \ldots, b_r, c, d_1, \ldots, d_t\}$ be a Mal'cev generating set for G. For each $\theta \in \mathrm{Aut}(G)$, there exist linear functions $f_1, \ldots, f_{n+r} \colon \mathbb{Z}^{n+r} \to \mathbb{Z}$, linear functions $g_0, g_1, \ldots, g_t \colon \mathbb{Z}^{t+1} \to \mathbb{Z}$, and quadratic functions $h_0, h_1, \ldots, h_t \colon \mathbb{Z}^{n+r} \to \mathbb{Z}$, such that for all Mal'cev normal form words $w = a_1^{i_1} \cdots a_n^{i_n} b_1^{i_{n+1}} \cdots b_r^{i_{n+r}} c^{q_0} d_1^{q_1} \cdots d_t^{q_t}$,*

$$\theta(w) = a_1^{f_1(i_1, \ldots, i_{n+r})} \cdots b_r^{f_{n+r}(i_1, \ldots, i_{n+r})} c^{g_0(q_0, \ldots, q_t) + h_0(i_1, \ldots, i_{n+r})} \cdots d_t^{g_t(q_0, \ldots, q_t) + h_t(i_1, \ldots, i_{n+r})}.$$

Proof. First note that the commutator subgroup is preserved by θ. Thus (taking $d_0 = c$ by convention), we have for all $\rho \in \{1, \ldots, n\}$, $\sigma \in \{1, \ldots, r\}$ and $\varsigma \in \{0, \ldots, t\}$,

$$\theta(a_\rho) = a_1^{A_{\rho 1}} \cdots a_n^{A_{\rho n}} b_1^{B_{\rho 1}} \cdots b_r^{B_{\rho n}} d_0^{D_{\rho 0}} d_1^{D_{\rho 1}} \cdots d_t^{D_{\rho t}}$$

$$\theta(b_\sigma) = a_1^{A_{(n+\sigma)1}} \cdots a_n^{A_{(n+\sigma)n}} b_1^{B_{(n+\sigma)1}} \cdots b_r^{B_{(n+\sigma)r}} d_0^{D_{(n+\sigma)0}} d_1^{D_{(n+\sigma)1}} \cdots d_t^{D_{(n+\sigma)t}}$$

$$\theta(d_\varsigma) = d_0^{D'_{\varsigma 0}} d_1^{D'_{\varsigma 1}} \cdots d_t^{D'_{\varsigma t}}.$$

for some $A_{11}, \ldots, A_{(n+r)n}, B_{11}, \ldots, B_{(n+r)r}, D_{10}, \ldots, D_{(n+r)t}, D'_{00}, \ldots, D'_{tt} \in \mathbb{Z}$. Let $g = \theta(a_1^{i_1} \cdots a_n^{i_n} b_1^{i_{n+1}} \cdots b_r^{i_{n+r}} d_0^{q_0} d_1^{q_1} \cdots d_t^{q_t})$. Then

$$g = \prod_{\rho=1}^{n+r} a_1^{i_\rho A_{\rho 1}} \cdots a_n^{i_\rho A_{\rho n}} b_1^{i_\rho B_{\rho 1}} \cdots b_r^{i_\rho B_{\rho n}} i_\rho d_0^{D_{\rho 0}} \cdots i_\rho d_m^{D_{\rho m}} \prod_{\varsigma=0}^{t} d_0^{q_\varsigma D'_{\varsigma 0}} \cdots d_t^{q_\varsigma D'_{\varsigma t}}$$

$$= a_1^{\sum\limits_{\rho=1}^{n+r} i_\rho A_{\rho 1}} \cdots b_r^{\sum\limits_{\rho=1}^{n+r} i_\rho B_{\rho r}}$$

$$\prod_{\varsigma=0}^{t} d_\varsigma^{\sum\limits_{\rho=1}^{n+r} i_\rho D_{\rho \varsigma} + \sum\limits_{m=0}^{t} q_m D'_{m\varsigma} + \sum\limits_{\substack{\rho,\rho'=1 \\ \rho<\rho'}}^{n} \sum\limits_{\substack{\sigma,\sigma'=1 \\ \sigma<\sigma'}}^{n+r} \pi_{\rho\rho'\varsigma} A_{\sigma\rho} A_{\sigma'\rho'} i_\sigma i_{\sigma'} + \sum\limits_{\rho=1}^{n} \sum\limits_{\rho'=1}^{r} \sum\limits_{\substack{\sigma,\sigma'=1 \\ \sigma<\sigma'}}^{n+r} \upsilon_{\rho\rho'\varsigma} A_{\sigma\rho} B_{\sigma'\rho'} i_\sigma i_{\sigma'}}$$

$$+ \sum\limits_{\substack{\rho,\rho'=1 \\ \rho<\rho'}}^{r} \sum\limits_{\substack{\sigma,\sigma'=1 \\ \sigma<\sigma'}}^{n+r} \tau_{\rho\rho'\varsigma} B_{\sigma\rho} B_{\sigma'\rho'} i_\sigma i_{\sigma'}$$

If $\sigma \in \{1, \ldots, n\}$, define $f_\sigma(i_1, \ldots, q_t) = \sum_{\rho=1}^{n+r} i_\rho A_{\rho\sigma}$, and if $\sigma \in \{n+1, \ldots, n+r\}$, let $f_\sigma(i_1, \ldots, q_t) = \sum_{\rho=1}^{n+r} i_\rho B_{\rho\sigma}$. For $\varsigma \in \{0, \ldots, t\}$, define functions g_ς and h_ς by

$$g_\varsigma(q_0, \ldots, q_t) = \sum_{m=0}^{t} q_m D'_{m\varsigma}$$

$$h_\varsigma(i_1, \ldots, i_{n+r}) = \sum_{\rho=1}^{n+r} i_\rho D_{\rho\varsigma} + \sum_{\substack{\rho,\rho'=1 \\ \rho<\rho'}}^{n} \sum_{\substack{\sigma,\sigma'=1 \\ \sigma<\sigma'}}^{n+r} \pi_{\rho\rho'\varsigma} A_{\sigma\rho} A_{\sigma'\rho'} i_\sigma i_{\sigma'} + \sum_{\rho=1}^{n} \sum_{\rho'=1}^{r} \sum_{\substack{\sigma,\sigma'=1 \\ \sigma<\sigma'}}^{n+r} \upsilon_{\rho\rho'\varsigma} A_{\sigma\rho} B_{\sigma'\rho'} i_\sigma i_{\sigma'}$$

$$+ \sum_{\substack{\rho,\rho'=1 \\ \rho<\rho'}}^{r} \sum_{\substack{\sigma,\sigma'=1 \\ \sigma<\sigma'}}^{n+r} \tau_{\rho\rho'\varsigma} B_{\sigma\rho} B_{\sigma'\rho'} i_\sigma i_{\sigma'}.$$

We have that $\theta(g)$ equals

$$a_1^{f_1(i_1, \ldots, i_{n+r})} \cdots b_r^{f_{n+r}(i_1, \ldots, i_{n+r})} d_0^{g_0(q_0, \ldots, q_t)+h_0(i_1, \ldots, i_{n+r})} \cdots d_t^{g_t(q_0, \ldots, q_t)+h_t(i_1, \ldots, i_{n+r})}.$$

Moreover, the functions f_σ and g_ς are linear, and the functions h_ς are quadratic, as required.

We generalise an equation in a group G to allow variables to be acted upon by automorphisms of G. As we will see in Lemma 4.4, solving twisted equations in G is 'equivalent' to solving equations in finite extensions of G.

Definition 4.2. Let G be a group. A *twisted equation* in G with *variables* V is an element $w \in (G \cup F(V) \times \mathrm{Aut}(G))^*$, and is again denoted $w = 1$. Define the function

$$p \colon G \times \mathrm{Aut}(G) \to G$$
$$(g, \ \psi) \mapsto g\psi.$$

If $\phi \colon F(V) \to G$ is a homomorphism, let $\bar{\phi}$ denote the (monoid) homomorphism from $(G \cup F(V) \times \mathrm{Aut}(G))^*$ to $(G \times \mathrm{Aut}(G))^*$, defined by $(h, \ \psi)\bar{\phi} = (h\phi, \ \psi)$ for $(h, \ \psi) \in F(V) \times \mathrm{Aut}(G)$ and $g\bar{\phi} = g$ for all $g \in G$. A *solution* to $w = 1$ is a homomorphism $\phi \colon F(V) \to G$, such that $w\bar{\phi}p = 1_G$.

For the purposes of decidability, in finitely generated groups, the elements of G will be represented as words over a finite generating set, and in twisted equations, automorphisms will be represented by their action on the generators.

The *single twisted equation problem* in G is the decidability question as to whether there is a terminating algorithm that accepts a twisted equation $w = 1$ as input, returns YES if $w = 1$ admits a solution and NO otherwise, where elements of G within w are represented by words over a finite generating set, and automorphisms are represented by their action on the finite generating set.

We give a brief example of a twisted equation in $\mathbb{Z}$.

Example 4.3. Consider the twisted equation $X = Y\psi$ in the group $\mathbb{Z}$ with the generator a, where $\psi \in \mathrm{Aut}(\mathbb{Z})$ maps a to a^{-1} (the unique non-identity automorphism). It follows that this equation is equivalent to $X = Y^{-1}$, which is not difficult to show has the solution set

$$\{(a^x, \ a^{-x}) \mid x \in \mathbb{Z}\}.$$

More generally, any twisted equation in $\mathbb{Z}$ can be solved using this argument, as the identity automorphism can simply be removed without affecting the solution set, and the non-identity automorphism can be replaced by adding an inverse sign to the variable it acts on. This yields an (untwisted) equation in $\mathbb{Z}$.

The following lemma is widely known, although often not stated explicitly. Variations of it have been used to show systems of equations in virtually free groups, or virtually abelian groups are decidable, or to describe the structure of solution sets (see for example [2, 3, 6]). We include a proof for completeness.

Lemma 4.4. *Let G be a group with a finite-index normal subgroup H, such that H has decidable single twisted equation problem. Then G has decidable single equation problem.*

Proof. Let T be a (finite) transversal for H. consider an equation $w = 1$ in G. We can express every element in G in the form ht for $h \in H$ and $t \in T$. Thus we can write $w = 1$ as

$$h_1 t_1 X_{i_1}^{\epsilon_1} \cdots h_K t_K X_{i_K}^{\epsilon_K} = 1, \qquad\qquad (4.1)$$

where $h_j \in H$, $t_j \in T$, and $\epsilon_j \in \{-1, \ 1\}$, for all j, and $X_1, \ \ldots, \ X_N$ are the variables of $= 1$. If $(g_1, \ \ldots, \ g_N)$ is a solution, then each g_j can be expressed in the normal subgroup-transversal form, and so by applying this fact to our variables, (4.1) admits a solution if

and only if the following equation does:

$$h_1 t_1 (Y_{i_1} Z_{i_1})^{\epsilon_1} \cdots h_K t_K (Y_K Z_K)^{\epsilon_K} = 1, \tag{4.2}$$

where $X_j = Y_j Z_j$, Y_j is a variable over H, and Z_j is a variable over T, for all j. For each $g \in G$, define $\psi_g \colon G \to G$ by $h\psi_g = ghg^{-1}$. As H is normal, these automorphisms fix H. We will abuse notation, and extend this notation to define $\psi_{Z_1}, \ldots, \psi_{Z_N}$. Let

$$\delta_j = \begin{cases} 0 & \epsilon_j = 1 \\ 1 & \epsilon_j = -1. \end{cases}$$

Thus (4.2) is equivalent to

$$(Y_{i_1}^{\epsilon_1} \psi_{Z_{i_1}}^{\delta_1}) Z_{i_1}^{\epsilon_1} h_1 t_1 \cdots (Y_{i_K}^{\epsilon_K} \psi_{Z_{i_K}}^{\delta_K}) Z_{i_K}^{\epsilon_K} h_K t_K = 1.$$

By pushing all Y_js and h_js to the left, we obtain

$$(Y_{i_1}^{\epsilon_1} \psi_{Z_{i_1}}^{\delta_1})(h_1 \psi_{Z_{i_1}}^{\epsilon_1}) \cdots (Y_{i_K}^{\epsilon_K} \psi_{Z_{i_K}}^{\delta_K} \psi_{t_{K-1}} \psi_{Z_{i_{K-1}}}^{\epsilon_{K-1}} \cdots \psi_{t_1} \psi_{Z_{i_1}}^{\epsilon_1})(h_K \psi_{Z_{i_K}}^{\epsilon_K} \psi_{t_{K-1}} \cdots \psi_{t_1} \psi_{Z_{i_1}}^{\epsilon_1}) Z_{i_1}^{\epsilon_1} t_1 \cdots Z_{i_K}^{\epsilon_K} t_K = 1. \tag{4.3}$$

We have that a necessary condition for a potential solution $(y_1 z_1, \ldots, y_N z_N)$ to (4.3) to be a solution is that $t_1 z_{i_1} \cdots t_K z_{i_K} \in H$. Let A be the set of tuples $(z_1, \ldots, z_N)$ of transversal elements such that $t_1 z_{i_1} \cdots t_K z_{i_K} \in H$. As T is finite, so is A, and so the solution set to (4.3) is equal to the finite union across $(z_1, \ldots, z_N) \in A$ of the following twisted equations in H:

$$(Y_{i_1}^{\epsilon_1} \psi_{z_{i_1}}^{\delta_1})(h_1 \psi_{z_{i_1}}^{\epsilon_1}) \cdots (Y_{i_K}^{\epsilon_K} \psi_{z_{i_K}}^{\delta_K} \psi_{t_{K-1}} \psi_{z_{i_{K-1}}}^{\epsilon_{K-1}} \cdots \psi_{t_1} \psi_{z_{i_1}}^{\epsilon_1})(h_K \psi_{z_{i_K}}^{\epsilon_K} \psi_{t_{K-1}} \cdots \psi_{t_1} \psi_{z_{i_1}}^{\epsilon_1}) z_{i_1}^{\epsilon_1} t_1 \cdots z_{i_K}^{\epsilon_K} t_K = 1$$

Since the twisted single equation problem in H is decidable, and we can check if each of these equations admit solutions, noting there are finitely many of them. If at least one admits a solution, then $w = 1$ does. If none admit a solution, then neither does $w = 1$.

Now that we have Lemma 4.4, the following is (almost) all that is required to prove that the single equation problem in a virtually class 2 nilpotent group with a virtually cyclic commutator subgroup is decidable.

Lemma 4.5. *The single twisted equation problem in a class 2 nilpotent group with a virtually cyclic commutator subgroup is decidable.*

Proof. Consider a single twisted equation $\mathcal{E}$ in a class 2 nilpotent group G. We can view $\mathcal{E}$ by applying the automorphisms to the words $\boldsymbol{\nu}_z$ within the statement of Lemma 3.3. Using Proposition 4.1, automorphisms act as linear functions of $\boldsymbol{\nu}_{z,a_1}, \ldots, \boldsymbol{\nu}_{z,a_n}$ and $\boldsymbol{\nu}_{z,b_1}, \ldots, \boldsymbol{\nu}_{z, b_r}$, and quadratic functions of $\boldsymbol{\nu}_{z, c}$ and $\boldsymbol{\nu}_{z, d_1}, \ldots, \boldsymbol{\nu}_{z, d_t}$.

In Lemma 3.3, the values $\boldsymbol{\nu}_{z, c}$ and $\boldsymbol{\nu}_{z, d_1}, \ldots, \boldsymbol{\nu}_{z, d_m}$ only appear in linear terms in the system (that is, they never appear in the form $\boldsymbol{\nu}_{z,c} \boldsymbol{\nu}_{z, d_i}$). Thus after applying the automorphisms, we will have a system of the form stated in Lemma 3.6 equivalent to $\mathcal{E}$. By Lemma 3.7, a system of the form of Lemma 3.6 is decidable, and thus the result follows.

All that remains to prove Theorem 1.1 is to deal with the difference between a finite-index subgroup and a finite-index normal subgroup.

Lemma 4.6. *Let N be a finite-index normal subgroup of a group H, such that H is nilpotent of class 2, and H has a virtually cyclic commutator subgroup. Then N is nilpotent of class 2 and has a virtually cyclic commutator subgroup.*

Proof. Subgroups of nilpotent groups of class c are always nilpotent (see, for example [1], Theorem 2.4), and thus N is nilpotent of class 2. Moreover $[N, N] \leq [H, H]$, and so $[N, N]$ is contained in a virtually cyclic group, and is therefore virtually cyclic.

Combining our lemmas now gives the following.

Theorem 1.1. *The single equation problem in virtually a group that is class* 2 *nilpotent group with a virtually cyclic commutator subgroup is decidable.*

Proof. Let $\mathcal{P}$ be the property of being class 2 nilpotent and having a virtually cyclic commutator subgroup. By taking the normal core, we have that a virtually $\mathcal{P}$ group admits a finite-index normal subgroup. Lemma 4.4 implies that this normal subgroup must be $\mathcal{P}$. We have therefore shown that any virtually $\mathcal{P}$ group has a finite-index normal subgroup that is $\mathcal{P}$. We have from Lemma 4.5 that the single twisted equation problem in a group with $\mathcal{P}$ is decidable. The result now follows by 4.6.

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